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Universal Galois Coverings of Self-Injective Algebras by Repetitive Algebras and Hochschild Cohomology

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1. INTRODUCTION

In this paper all algebras are basic, connected, finite-dimensional algebras over an algebraically closed field k. For an algebra A we denote by mod A the category of finitely generated right A-modules and by D: mod $A \to$ mod A^{op} the standard duality $\mathrm{Hom}_k(\cdot, k)$.

We are interested in studying the representation theory of self-injective algebras. An algebra A is called *self-injective* if $A \simeq D(A)$ in mod A, that is, if the projective A-modules are injective. Examples of self-injective algebras are provided by algebras of finite groups. Frequently, self-injective algebras have triangular Galois coverings, and then we may reduce the study of such algebras and their representations to that for the corresponding algebras of finite global dimension. This is the case for all representation–finite self-injective algebras [5, 15], certain classes of tame representation–infinite self-injective algebras [2, 3, 8, 10, 20], and certain classes of wild representation–infinite self-injective algebras [9]. There is a close connection between self-injective algebras and repetitive algebras [15]. We recall that if B is an algebra of finite global dimension and $\widehat{B}$ is the repetitive algebra of B, then the stable module category $\underline{\mathrm{mod}}\,\widehat{B}$ of $\mathrm{mod}\,\widehat{B}$ is equivalent, as a triangulated category, to the derived category $D^b(\mathrm{mod}\,B)$ of bounded complexes over $\mathrm{mod}\,B$ (see [13]).

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We are interested in a recent class of self-injective algebras, namely the *repetitive covering self-injective algebras*, studied by Skowroński and Yamagata in [21, 22]. Examples of these are provided by trivial extensions.

DEFINITION. A basic, connected, self-injective finite-dimensional algebra over an algebraically closed field k is called a *repetitive covering self-injective algebra* if there is an ideal I of A such that the ordinary quiver Q_B of $B = A/I$ has no oriented cycles, $IeI = 0$, and Ie is an injective cogenerator in $\operatorname{mod} B$, where e is a residual identity of B.

Given a *repetitive covering self-injective algebra* A, Skowroński and Yamagata [22] proved that the repetitive algebra $\widehat{B}$ of B is a Galois covering of A with an infinite cyclic Galois group G generated by an automorphism $\varphi\nu_{\widehat{B}}$, where φ is a positive automorphism of $\widehat{B}$ and $\nu_{\widehat{B}}$ is the Nakayama shift of $\widehat{B}$. In particular, if $G = (\nu_{\widehat{B}})$ then $\widehat{B}/(\nu_{\widehat{B}})$ is isomorphic to the trivial extension $T(B) = B \ltimes D(B)$ of B by $D(B)$.

Recall that an algebra B is called *triangular* if its ordinary quiver Q_B has no oriented cylces, and it is called *schurian* if $\dim_k \operatorname{Hom}_B(P, P') \leq 1$ for every pair of indecomposable projective B-modules. Triangular representation–finite algebras are always schurian.

The purpose of this paper is the following: finding conditions for the repetitive algebra of B to be the universal Galois covering of A, and deducing an estimation of $H^1(A)$, the first Hochschild cohomology group of A with coefficients in A, under the assumptions that B is triangular and schurian.

Recall that Galois coverings are obtained by means of fundamental groups [17]. The following is our first main result.

THEOREM A. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group G, and let B be schurian. Let (Q_A, I_A) be any presentation of A associated to the presentation (Q_B, I_B) of B. Then we have the following short exact sequence of groups*:

$$1 \to \pi_1(Q_B, I_B) \to \pi_1(Q_A, I_A) \to G \to 1.$$

As an immediate application we have that $F\colon \widehat{B} \to A$ is the universal Galois covering of A if and only if B is simply connected. In particular, if $G = (\nu_{\widehat{B}})$, we have that $\widehat{B} \to T(B) = B \ltimes D(B)$ is the universal Galois covering of the trivial extension $T(B)$ of B by $D(B)$ if and only if B is simply connected.

Concerning the first Hochschild cohomology group of a repetitive covering self-injective algebra A, we first show that $H^1(A)$ is nonzero. We are now in a position to formulate our second main result.

THEOREM B. *Let A be a repetitive covering self-injective algebra, let $F: \widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu^c_{\widehat{B}})$, let $c \neq 0$, and let B be schurian. Then*

$$0 \to \operatorname{Hom}(\pi_1(Q_B, I), k^+) \to H^1(A) \to k^+ \to 0$$

is an exact sequence of abelian groups.

As an immediate consequence we have that if B is simply connected then $H^1(A) = k$.

The paper is organized as follows. In Section 2 we describe all of the presentations of the repetitive algebra $\widehat{B}$ for any triangular schurian algebra B. In Section 3 we prove Theorem A and its applications to the universal Galois covering of repetitive covering self-injective algebras. Finally, Section 4 is devoted to the connection between the first Hochschild cohomology group of $A \simeq \widehat{B}/\langle \nu^c_{\widehat{B}} \rangle$ and the fundamental group of B.

2. PRELIMINARIES

2.1. *Galois Coverings and Repetitive Covering Self-Injective Algebras*

Recall that a k-category R is a category such that the set of morphisms $R(e_x, e_y)$ from e_x to e_y has a k-vector space structure for each pair of objects e_x, e_y in R, and such that the composition of morphisms is k-bilinear. A k-category R is called locally bounded if (a) $R(e_x, e_x)$ is local for each object e_x in R; (b) distinct objects are not isomorphic; and (c) $\sum_{e_y \in R} \dim_k R(e_x, e_y) < \infty$ and $\sum_{e_y \in R} \dim_k R(e_y, e_x) < \infty$ for each object e_x in R.

We may consider a basic algebra A as a k-category with the object set consisting of a fixed complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ and sets of morphisms $A(e_x, e_y) = e_y A e_x$.

Locally bounded categories can be viewed as locally finite quivers. Recall that a *quiver* Q is defined by a set of points Q_0, a set of arrows Q_1, and two maps s, t: $Q_1 \to Q_0$ associating with each arrow its starting and terminal points, respectively. We denote by kQ the path algebra of Q. We write a path α in kQ as a composition of consecutive arrows $\alpha = \alpha_1 \cdots \alpha_r$, where $t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \ldots, r-1$, and we set $s(\alpha) = s(\alpha_1)$, $t(\alpha) = t(\alpha_r)$. If R is a locally bounded category then $R = kQ_R/I_R$, where (a) Q_R is locally finite, that is, the number of arrows starting or ending at any vertex is finite, and (b) for each vertex $x \in (Q_R)_0$ there is a natural number N_x such that I_R contains each path of length $\geq N_x$ starting or ending at x. The pair (Q_R, I_R) is called a *presentation* of R.

In particular, if R is a locally bounded k-category, the *radical* $\operatorname{rad} R$ of R is the ideal assigning to a pair of objects (e_x, e_y) the subspace $\operatorname{rad} R(e_x, e_y)$

of $R(e_x, e_y)$ consisting of the noninvertible morphisms. The *radical square* $\operatorname{rad}^2 R$ is defined by $\operatorname{rad}^2 R(e_x, e_y) = \sum_{e_z \in R} \operatorname{rad} R(e_z, e_y) \operatorname{rad} R(e_x, e_z)$. The set of vertices of the quiver Q_R of R is $\{x : e_x \text{ in } R\}$, and if x, y are two such vertices the number of arrows $y \to x$ is equal to $\dim_k \operatorname{rad} R(e_x, e_y)/\operatorname{rad}^2 R(e_x, e_y)$. We get an isomorphism $kQ_R/I_R \xrightarrow{\sim} R$ for some admissible ideal I_R by sending the arrows $y \to x$ onto representatives in $\operatorname{rad} R(e_x, e_y)$ of chosen basis vectors of $\operatorname{rad} R(e_x, e_y)/\operatorname{rad}^2 R(e_x, e_y)$.

Let R and A be locally bounded k-categories. A *Galois covering* $F\colon R \to A$ defined by the action of the group G is a functor satisfying

(i) G is a group of k-linear automorphisms acting freely on the objects of R (that is, $ge_x = e_x$ implies $g = 1$);

(ii) $Fg = F$ for every $g \in G$;

(iii) F is onto on objects and G acts transitively on $F^{-1}(e_a)$ for every object e_a in A;

(iv) F is a *covering functor*, that is, the induced maps

$$\bigoplus_{F(e_y)=e_a} R(e_x, e_y) \to A(F(e_x), e_a) \quad \text{and} \quad \bigoplus_{F(e_y)=e_a} R(e_y, e_x) \to A(e_a, F(e_x))$$

are bijective for all e_x in R and e_a in A.

Remark 2.1. If $F\colon R \to A$ is a covering functor then

$$\bigoplus_{F(e_y)=e_b} \operatorname{rad} R(e_x, e_y)/\operatorname{rad}^2 R(e_x, e_y) \xrightarrow{\sim} \operatorname{rad} A(F(e_x), e_b)/\operatorname{rad}^2 A(F(e_x), e_b).$$

In particular, if the quiver Q_R contains an arrow $y \to x$, then Q_A contains an arrow $b \to a$, where $e_a = F(e_x)$. In other words, F induces a quiver morphism $Q_F\colon Q_R \to Q_A$ such that the image of the admissible ideal I_R of kQ_R is contained in I_A (see [4]). Moreover, if F is a Galois covering then Q_F is onto.

A Galois covering $F\colon R \to A$ given by the action of the group G is called the *universal Galois covering* if for any Galois covering $F'\colon R' \to A$ given by the action of the group G', there exists a unique Galois covering $H\colon R \to R'$ given by the action of a normal subgroup N of G such that $G/N \simeq G'$ and $F = F'H$.

Let (Q_A, I_A) be a presentation of A, Q_A without double arrows. The universal Galois covering $F\colon k\widetilde{Q}/\widetilde{I} \to kQ_A/I_A$ is constructed in [17] in the following way: for an arrow α in Q_A we denote by α^{-1} its formal inverse. A walk in Q_A is a formal composition $\alpha_1^{\epsilon_1} \cdots \alpha_t^{\epsilon_t}$ with $\alpha_i \in (Q_A)_1$ and $\epsilon_i = \pm 1$ for any i with $1 \le i \le t$. We denote by $\sim$ the equivalence relation defined on the set of all walks in Q_A induced by the elementary relations

(a) $\alpha^{-1}\alpha \sim e_b$ and $\alpha\alpha^{-1} \sim e_a$ if $\alpha\colon a \to b$ is an arrow in Q_A;

(b) $u_i \sim u_j$ for every $1 \le i, j \le m$ if $\rho = \sum_{i=1}^m \lambda_i u_i \in I_A(a,b)$ is a *minimal relation*, that is, $m \ge 2$ and $\sum_{i\in J} \lambda_i u_i \notin I_A(a,b)$ for all $\emptyset \ne J \subsetneq \{1,\ldots,m\}$;

(c) $wuw' \sim wvw'$ if $u \sim v$ and the product makes sense.

Let W be the set of all walks in Q_A starting at a fixed vertex x_0. Then $\widetilde{Q}_0 = W/\sim$ is the set of vertices of a quiver $\widetilde{Q}$ with arrows $[w] \xrightarrow{\alpha} [w\alpha]$ for any $\alpha \in (Q_A)_1$. Let $\pi\colon \widetilde{Q} \to Q_A$ be the natural projection sending a vertex $\tilde{x}$ of $\widetilde{Q}$ to the terminus of any representative of $\tilde{x}$. Let $\widetilde{I}$ be the admissible ideal of $k\widetilde{Q}$ generated by the liftings through π of the zero and the minimal relations in I_A. The induced functor $F\colon k\widetilde{Q}/\widetilde{I} \to kQ_A/I_A$ is a Galois covering defined by the action of the *fundamental group* $\pi_1(Q_A, I_A)$ consisting of the equivalence classes of closed walks with origin x_0.

Remark 2.2. Let $F\colon R \to A$ be a Galois covering and let $Q_F\colon Q_R \to Q_A$ be the induced quiver morphism. If $\rho = \sum_{i=1}^m \lambda_i u_i \in I_A(a,b)$ is a minimal relation then ρ belongs to the image of the admissible ideal I_R. More precisely, for $e_a = F(e_x)$ consider the isomorphism

$$\bigoplus_{F(e_y)=e_b} R(e_x, e_y) \to A(F(e_x), e_b),$$

and let $\overline{u_i} = F(\overline{w_i})$, where $\overline{w_i} \in \sum_j R(e_x, e_{y_{i_j}})$ with $F(e_{y_{i_j}}) = e_b$ for each $i = 1,\ldots,m$. Then $\sum_{i=1}^m \lambda_i w_i \in I_R$ and $w_i \notin I_R$ for all $1 \le i \le m$.

We refer to [4, 12, 17] for more details on covering functors.

Let A be a self-injective algebra, and let $\{e_1,\ldots,e_n\}$ be a complete set of primitive orthogonal idempotents of A. Then $\{e_1A,\ldots,e_nA\}$ is a complete set of nonisomorphic indecomposable projective A-modules and $\{D(Ae_1),\ldots,D(Ae_n)\}$ is a complete set of nonisomorphic indecomposable injective A-modules. Let ν be the *Nakayama automorphism* of A where $\theta_r\colon A \to D(A)$ is an A-bimodule isomorphism such that $\theta_r(ab) = \theta_r(a)b = \nu(a)\theta_r(b)$. Since $\{\nu(e_1)A,\ldots,\nu(e_n)A\}$ is a complete set of nonisomorphic indecomposable projective A-modules, there is a permutation of $\{1,\ldots,n\}$, denoted again by ν, such that $\nu(e_i)A \simeq e_{\nu(i)}A$ for all $i \in \{1,\ldots,n\}$. By the Krull–Schmidt theorem we may assume that $\nu(e_iA) = \nu(e_i)A = e_{\nu(i)}A$ for all $i \in \{1,\ldots,n\}$.

Assume that I is a two-sided ideal of A, $B = A/I$, and e is an idempotent of A such that $e + I$ is an identity of B. We may assume that $e = e_1 + \cdots + e_t$ for some $t \le n$, and that $\{e_1,\ldots,e_t\}$ is the subset of $\{e_1,\ldots,e_n\}$ consisting of all idempotents which are not in I. Note that $B \simeq eAe/eIe$ and $1 - e \in I$. The idempotent e is uniquely determined by I up to an inner automorphism of A, and we call it a *residual identity* of B.

There is a retraction ι of the canonical algebra homomorphism $\rho\colon eAe \to eAe/eIe$ such that $\iota(e_i) = e_i$ for any i with $1 \le i \le t$ (see [22]). Furthermore, B and eAe/eIe can be considered as A-algebras by the canonical algebra homomorphisms $A \to B \to eAe/eIe$. Hence B, eAe/eIe, and $\iota(B)$ are isomorphic A-algebras, and from now on we shall identify them.

Consider the repetitive algebra

$$\widehat{B} = \begin{pmatrix} \ddots & \ddots & & & 0 \\ & B_{m-1} & Q_{m-1} & & \\ & & B_m & Q_m & \\ & & & B_{m+1} & \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

of the algebra B where $B_m = B$ and $Q_m = D(B)$ for all $m \in \mathbf{Z}$, all of the remaining entries are zero, and the matrices in $\widehat{B}$ have only finitely many nonzero elements. Addition is the usual addition of matrices, and multiplication is induced from the canonical maps $B \otimes_B D(B) \to D(B)$, $D(B) \otimes_B B \to D(B)$ and the zero map $D(B) \otimes_B D(B) \to 0$.

We may clearly consider $\widehat{B}$ as a locally bounded k-category with the object set consisting of the set of primitive orthogonal idempotents $\{e_{m,i}\}_{1\le i\le t,\, m\in\mathbf{Z}}$ and morphisms

$$\widehat{B}(e_{s,j}, e_{m,i}) = e_{m,i}\widehat{B}e_{s,j} = \begin{cases} e_iBe_j & \text{if } s = m, \\ e_iD(B)e_j & \text{if } s = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

The canonical shifting automorphism $\nu_{\widehat{B}}\colon \widehat{B} \to \widehat{B}$ with $\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for all $m \in \mathbf{Z}$, $1 \le i \le t$, is called the Nakayama automorphism of $\widehat{B}$. An automorphism φ of $\widehat{B}$ is said to be positive if for each object $e_{m,i}$ of $\widehat{B}$ there exist p, j with $p \ge m$ and $1 \le j \le t$, such that $\varphi(e_{m,i}) = e_{p,j}$. Hence a positive automorphism of $\widehat{B}$ is a shift of $\widehat{B}$ in the same direction as $\nu_{\widehat{B}}$.

DEFINITION 2.3. A basic, connected, self-injective, finite-dimensional algebra over an algebraically closed field k is called a *repetitive covering self-injective algebra* if there is an ideal I of A such that the ordinary quiver Q_B of $B = A/I$ has no oriented cycles, $IeI = 0$, and Ie is an injective cogenerator in $\operatorname{mod} B$, where $e = e_1 + \cdots + e_t$ is a residual identity of B.

THEOREM 2.4 [22, Theorem 4.1]. *Let A be a repetitive covering self-injective algebra. Then there is a Galois covering $F\colon \widehat{B} \to A$, where the Galois group G is an infinite cyclic group of automorphisms of $\widehat{B}$ generated by $\varphi\nu_{\widehat{B}}$ for some positive automorphism φ of $\widehat{B}$.*

The functor $F\colon \widehat{B} \to A$ is defined as follows: $F(e_{m,i}) = e_{\nu^m(i)}$ and $F\colon \widehat{B}(e_{s,j}, e_{m,i}) \to A(F(e_{s,j}), F(e_{m,i})) = A(e_{\nu^s(j)}, e_{\nu^m(i)})$ is given by

(a) if $s = m$, F is the composition

$$e_{m,i}\widehat{B}e_{m,j} = e_i B e_j \hookrightarrow e_i A e_j \xrightarrow{\nu^m} e_{\nu^m(i)} A e_{\nu^m(j)};$$

(b) if $s = m+1$, F is the composition

$$e_{m,i}\widehat{B}e_{m+1,j} = e_i D(B) e_j \xrightarrow{\theta^{-1}} e_i I e_{\nu(j)} \hookrightarrow e_i A e_{\nu(j)} \xrightarrow{\nu^m} e_{\nu^m(i)} A e_{\nu^{m+1}(j)};$$

(c) F is zero for $s \neq m$ and $s \neq m+1$,

where the isomorphism θ is induced by the isomorphism $\theta_l\colon A \to D(A)$ such that $\theta_l(ab) = \theta_l(a)\nu^{-1}(b) = a\theta_l(b)$ (see [22]).

The group G is generated by an automorphism $\varphi\nu_{\widehat{B}}$, where φ is a positive automorphism of $\widehat{B}$ such that $g = (\varphi\nu_{\widehat{B}})^{-1}$ is defined on the objects by $g(e_{m,i}) = e_{m-\eta(i),\, \nu^{\eta(i)}(i)}$, where $\eta(i) = \min\{k > 0 : \nu^k(i) \leq t\}$. The action of g on $\widehat{B}(e_{s,j}, e_{m,i})$ is defined by the map

$$F_2^{-1}F_1\colon \widehat{B}(e_{s,j}, e_{m,i}) \to \widehat{B}(g(e_{s,j}), g(e_{m,i})),$$

where

$$F_1\colon \widehat{B}(e_{s,j}, e_{m,i}) \to F(\widehat{B}(e_{s,j}, e_{m,i})),$$

$$F_2\colon \widehat{B}(g(e_{s,j}), g(e_{m,i})) \to F(\widehat{B}(g(e_{s,j}), g(e_{m,i})))$$

are the k-linear maps induced by F, monomorphisms by definition, whose images are equal [22, p. 725].

From now on, whenever we consider a Galois covering $F\colon \widehat{B} \to A$ we will assume that it has the property given in Theorem 2.4.

To give some examples, we recall the following result proved by Skowroński and Yamagata.

Proposition 2.5 [21, Proposition 2.3]. *Let I be an ideal of A, $B = A/I$, let e be a residual identity of B, and assume that $IeI = 0$. Then the following conditions are equivalent*:

(i) *Ie is an injective cogenerator in* mod B;

(ii) $Ie = l_A(I) = \{a \in A : aI = 0\}$.

Example 2.6. The trivial extension $A = B \ltimes D(B)$ of B by $D(B)$ is a repetitive covering self-injective algebra, the Nakayama automorphism ν of A is the identity, and $I = D(B)$. Recall that $B \ltimes D(B)$ is the symmetric algebra whose underlying vector space is $B \oplus D(B)$, and the product is defined by $(a, f) \cdot (b, g) = (ab, ag + fb)$ for any $a, b \in B$, $f, g \in D(B)$. Moreover, the natural functor $\widehat{B} \to B \ltimes D(B) = A$ is the well-known Galois covering with the Galois group generated by $\nu_{\widehat{B}}$.

EXAMPLE 2.7. Let $A = kQ/J^{t+1}$, where Q is the oriented cycle with n vertices and n arrows, $1 \leq t \leq n$, and J is the ideal generated by the arrows. Then A is a repetitive covering self-injective algebra, where $B = kQ_B$ is the hereditary algebra with

$$Q_B\colon 1 \to 2 \to \cdots \to t.$$

If $t = n$ then $A \simeq B \ltimes D(B)$. If $1 \leq t < n$ take $e = \sum_{i=1}^{t} e_i$ and consider the ideal $I = A(1-e)A$. Then $IeI = 0$, and Ie is an injective cogenerator in $\operatorname{mod} B$ since $B = A/I$ and $Ie = l_A(I)$. Moreover, $A \simeq \widehat{B}/G$ with G generated by the automorphism γ^n, where $\gamma\colon \widehat{B} \to \widehat{B}$ is given by $\gamma(m,i) = (m, i+1)$ if $1 \leq i \leq t-1$ and $\gamma(m,t) = (m+1, 1)$.

EXAMPLE 2.8. Given a triangular algebra B, let $A = \widehat{B}/(\nu_{\widehat{B}}^{c})$, $c \neq 0$. Then A is a repetitive covering self-injective algebra: if $c = 1$ then $A \simeq B \ltimes D(B)$ is the trivial extension of B by $D(B)$; if $c > 1$ consider the ideal $I = A(1-e)A$ with $e = \sum_{i \in (Q_B)_0} \overline{e_{0,i}}$. Then $IeI = 0$, and Ie is an injective cogenerator in $\operatorname{mod} B$ since $B = A/I$ and $Ie = l_A(I)$. In a similar way, it can be seen that the following representatives of the derived equivalence classes of standard representation–finite self-injective algebras are repetitive covering self-injective algebras: $\Lambda(A_n, s/n, 1)$, $s \geq n$, $\Lambda(A_{2p+1}, s, 2)$, $\Lambda(D_n, s, 1)$, $\Lambda(D_n, s, 2)$, $\Lambda(D_4, s, 3)$, $\Lambda(E_n, s, 1)$, and $\Lambda(E_6, s, 2)$ (see [1, 2.5]).

EXAMPLE 2.9. Let A be a self-injective algebra whose Auslander–Reiten quiver Γ_A admits a nonperiodic generalized standard right (left) stable full translation subquiver which is closed under successors (predecessors) in Γ_A. Then A is a repetitive covering self-injective algebra, $A \simeq \widehat{B}/(\varphi\nu_{\widehat{B}})$, where B is a tilted algebra not of Dynkin type, $B = \operatorname{End}_H(T)$ for some hereditary algebra H and a tilting H-module T without nonzero preprojective (preinjective) direct summands, and φ is a positive automorphism of $\widehat{B}$ (see [22, Theorem 5.5]).

The functor $F\colon \widehat{B} \to A$ is the universal Galois covering of A if $\pi_1(Q_A, I_A) \simeq G$ for any presentation (Q_A, I_A) of A. So we want to study the relationship between the fundamental groups $\pi_1(Q_A, I_A)$ and $\pi_1(Q_B, I_B)$. To do this, we need the description of the presentations of A and B.

2.2. *Presentations of Repetitive Algebras*

We are going to describe the presentations of the locally bounded k-category $\widehat{B}$ for any triangular schurian algebra B.

Let (Q_B, I_B) be a fixed presentation of B. Given γ, a path in kQ_B, we denote by $\bar{\gamma}$ the corresponding element in kQ_B/I_B.

Denote $r = \operatorname{rad}(B)$. Since $D(B)^2 = 0$ in $\widehat{B}$, we have

$$\operatorname{rad}\widehat{B}(e_{s,j}, e_{m,i}) = \begin{cases} e_i r e_j & \text{if } s = m, \\ e_i D(B) e_j & \text{if } s = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned}\operatorname{rad}^2\widehat{B}(e_{m,j}, e_{m,i}) &= \sum_{e_{l,t}\in\widehat{B}} \operatorname{rad}\widehat{B}(e_{l,t}, e_{m,i})\operatorname{rad}\widehat{B}(e_{m,j}, e_{l,t}) \\ &= \sum_{e_{m,t}\in\widehat{B}} \operatorname{rad}\widehat{B}(e_{m,t}, e_{m,i})\operatorname{rad}\widehat{B}(e_{m,j}, e_{m,t}) \\ &= \sum_{e_t\in B} (e_i r e_t)(e_t r e_j) = e_i r^2 e_j,\end{aligned}$$

and

$$\begin{aligned}\operatorname{rad}^2\widehat{B}(e_{m+1,j}, e_{m,i}) &= \sum_{e_{l,t}\in\widehat{B}} \operatorname{rad}\widehat{B}(e_{l,t}, e_{m,i})\operatorname{rad}\widehat{B}(e_{m+1,j}, e_{l,t}) \\ &= \sum_{e_{m,t}\in\widehat{B}} \operatorname{rad}\widehat{B}(e_{m,t}, e_{m,i})\operatorname{rad}\widehat{B}(e_{m+1,j}, e_{m,t}) \\ &\quad + \sum_{e_{m+1,t}\in\widehat{B}} \operatorname{rad}\widehat{B}(e_{m+1,t}, e_{m,i})\operatorname{rad}\widehat{B}(e_{m+1,j}, e_{m+1,t}) \\ &= \sum_{e_t\in B} (e_i r e_t)\cdot(e_t D(B) e_j) + (e_i D(B) e_t)\cdot(e_t r e_j) \\ &= e_i(D(B)r + rD(B))e_j.\end{aligned}$$

LEMMA 2.10 [11].

$$0 \to D(B)r + rD(B) \to D(B) \to D(\operatorname{soc}_{B^e} B) \to 0$$

is a short exact sequence of B-bimodules, where $B^e = B \otimes_k B^{op}$ *is the enveloping algebra.*

Proof. It is known that mod B^e is equivalent to the category of finitely generated B-bimodules, and

$$0 \to \operatorname{rad}_{B^e} D(B) \to D(B) \to D(\operatorname{soc}_{B^e} B) \to 0$$

is a short exact sequence of B^e-modules, so we only have to describe $\operatorname{rad}_{B^e} D(B)$. Since $\operatorname{rad}(B^e) = r \otimes B^{op} + B \otimes r^{op}$, we know that $\operatorname{rad}_{B^e} D(B) = \operatorname{rad} B^e\ D(B) \simeq rD(B) + D(B)r$, and the proof is complete. ∎

DEFINITION 2.11. A path q in kQ_B is called *maximal* if $\bar{q} \neq 0$ and $\overline{\alpha q} = 0 = \overline{q\alpha}$ in B for any arrow $\alpha \in (Q_B)_1$.

COROLLARY 2.12. *Let B be a schurian algebra. Then $D(B)/(rD(B) + D(B)r)$ is isomorphic as a B-bimodule to the dual of the subspace of B generated by all $\bar{q} \in B$ with q maximal in Q_B.*

Proof. It is well known that $\mathrm{soc}_{B^e}(B) = \mathrm{ann}_B(\mathrm{rad}\, B^e)$, so, by the previous lemma, $D(B)/(D(B)r + rD(B)) \simeq D(\{b \in B : (r \otimes B^{op} + B \otimes r^{op})b = 0\}) = D(\{b \in B : ab = 0 = ba \quad \text{for all } a \in r\})$. But, since B is schurian, $\{b \in B : ab = 0 = ba \quad \text{for all } a \in r\}$ is the subspace generated by the set $\{\bar{q} \in B : q \text{ a path in } kQ_B,\ a\bar{q} = 0 = \bar{q}a \quad \text{for all } a \in r\} = \{\bar{q} \in B : q \text{ a path in } kQ_B,\ \overline{\alpha q} = 0 = \overline{q\alpha} \quad \text{for all } \alpha \in (Q_B)_1\}$. This subset of B consists of all $\bar{q} \in B$ with q maximal in Q_B. ∎

PROPOSITION 2.13. *If B is a schurian algebra with ordinary quiver Q_B then the ordinary quiver of $\widehat{B}$ is given by*

$$(Q_{\widehat{B}})_0 = \{(m, i)\}_{m \in \mathbf{Z},\, i \in (Q_B)_0}$$

and

$$(Q_{\widehat{B}})_1 = \{\alpha^m : (m, j) \to (m, i)\}_{\alpha:\ j \to i \in (Q_B)_1,\, m \in \mathbf{Z}} \cup \{\beta^m_{p_1}, \dots, \beta^m_{p_r}\}_{m \in \mathbf{Z}},$$

where $p_1, \dots, p_r$ are maximal and $\{\overline{p_1}, \dots, \overline{p_r}\}$ is a basis of the subspace of B generated by all $\bar{q} \in B$ such that q is maximal, and $\beta^m_{p_i}$ is an arrow from $(m, t(p_i))$ to $(m+1, s(p_i))$ for each i.

Proof. The vertices of the quiver $Q_{\widehat{B}}$ are the objects of $\widehat{B}$, so $(Q_{\widehat{B}})_0 = \{(m, i)\}_{m \in \mathbf{Z},\, i \in (Q_B)_0}$. For each pair of vertices (m, i), (s, j), the number of arrows from (m, i) to (s, j) is equal to

$$\dim_k \mathrm{rad}\, \widehat{B}(e_{s,j}, e_{m,i})/\mathrm{rad}^2\, \widehat{B}(e_{s,j}, e_{m,i}).$$

This is zero if $s \neq m, m+1$. If $s = m$, it is equal to $\dim_k e_i(r/r^2)e_j$. If $s = m+1$, it is equal to $\dim_k (e_i D(B) e_j)/(e_i(D(B)r + rD(B))e_j) = \dim_k e_i(\overline{D}(B)/(D(B)r + rD(B)))e_j$, which is zero or one, depending on the existence of a maximal path in kQ_B from j to i (see Corollary 2.12). ∎

From now on, we are going to denote by $\mathbf{P} = \{p_1, \dots, p_r\}$ a set of maximal paths in kQ_B such that $\{\overline{p_1}, \dots, \overline{p_r}\}$ is a basis of the subspace of B generated by all $\bar{q} \in B$ such that q is maximal.

Given a basis $\{\overline{q_1}, \dots, \overline{q_s}\}$ of B, we denote by $\{\overline{q_1}^*, \dots, \overline{q_s}^*\}$ its dual basis in $D(B)$, where $\overline{q_i}^*(\overline{q_j})$ is 1 if $i = j$ and 0 otherwise.

Now we will describe all of the presentations $(Q_{\widehat{B}}, I_{\widehat{B}})$ of $\widehat{B}$ associated to a fixed presentation (Q_B, I_B) of B.

Let ϕ: $kQ_{\widehat{B}} \to \widehat{B}$ be a surjective morphism of algebras such that the images $\phi(\alpha)$ of the arrows α: $(m, i) \to (s, j)$ of $Q_{\widehat{B}}$ induce a basis of $\mathrm{rad}\, \widehat{B}(e_{s,j}, e_{m,i})/\mathrm{rad}^2\, \widehat{B}(e_{s,j}, e_{m,i})$. The ideal $I_{\widehat{B}}$ of $kQ_{\widehat{B}}$ such that

$kQ_{\widehat{B}}/I_{\widehat{B}} \simeq \widehat{B}$ is admissible. Now, since Q_B has no oriented cycles and B is schurian, we have

$$\phi(\alpha^m) \in \widehat{B}(e_{m,\,t(\alpha)}, e_{m,\,s(\alpha)}) = e_{s(\alpha)}Be_{t(\alpha)} = \{\lambda\alpha, \lambda \in k\}$$

and

$$\phi(\beta^m_{p_i}) \in \widehat{B}(e_{m+1,\,s(p_i)}, e_{m,\,t(p_i)}) = e_{t(p_i)}D(B)e_{s(p_i)}.$$

So $\phi(\alpha^m) = \lambda_m\alpha^m$, $0 \neq \lambda_m \in k$, and $\phi(\beta^m_{p_i}) = f^m_i$, where $\tilde{f}^m_i$ generates the k-vector space $e_{t(p_i)}(D(B)/(rD(B) + D(B)r))e_{s(p_i)}$. Hence the set $\{f^m_1, \ldots, f^m_r\}$ may be extended to a basis $\{f^m_1, \ldots, f^m_r, f^m_{r+1}, \ldots, f^m_s\}$ of $D(B)$. Therefore there exists a basis $\{\overline{q^m_1}, \ldots, \overline{q^m_s}\}$ of B whose dual is the given basis, that is, $f^m_i = \overline{q^m_i}^*$. From the fact that $\tilde{\bar{q}}^* \neq 0$ in $D(B)/(rD(B) + D(B)r)$ if and only if $\bar{q}$ is maximal (see Corollary 2.12), it follows that $\{\overline{q^m_1}, \ldots, \overline{q^m_r}\}$ is a basis of the subspace of B generated by all $\bar{q} \in B$ such that q is maximal. Since B is schurian, by reordering the given basis if necessary, we have $\overline{q^m_i} = \mu^m_i\overline{p_i}$ for all $p_i \in \mathbf{P}$ and some $0 \neq \mu^m_i \in k$. So $\phi(\beta^m_{p_i}) = \overline{q^m_i}^* = \mu^m_i\overline{p_i}^*$ for some $0 \neq \mu^m_i \in k$.

To describe the fundamental group we need the description of the minimal relations in $I_{\widehat{B}}$. Since the nonzero coefficients of these minimal relations do not matter in the definition of the equivalence relation used to define the fundamental group (see Section 2.1), we may assume without loss of generality that $\phi(\alpha^m) = \bar{\alpha}^m$ and $\phi(\beta^m_{p_i}) = \overline{p_i}^*$.

Notation. Given any path $\gamma = \alpha_1 \cdots \alpha_t$ in kQ_B, we denote by $\gamma^m = \alpha^m_1 \cdots \alpha^m_t$ the corresponding paths in $kQ_{\widehat{B}}$ and by $\bar{\gamma}^m = \overline{\alpha_1 \cdots \alpha_t}^m = \phi(\alpha^m_1 \cdots \alpha^m_t)$ the corresponding elements in $\widehat{B}$.

To describe the corresponding ideal $I_\phi = \operatorname{Ker}\phi$ we need the following definitions.

DEFINITION 2.14. A path C in $kQ_{\widehat{B}}$ is called *elementary* if

$$C = \alpha^m_s \cdots \alpha^m_t\beta^m_p\alpha^{m+1}_1 \cdots \alpha^{m+1}_{s-1}$$

with $\overline{\alpha_1 \cdots \alpha_t} = \mu\bar{p}$ for p a maximal path in $\mathbf{P}$ and $0 \neq \mu \in k$.

DEFINITION 2.15. A path q in $kQ_{\widehat{B}}$ is said to have a *supplement* if there exists a path q' in $kQ_{\widehat{B}}$ such that qq' is an elementary path in $kQ_{\widehat{B}}$.

Recall that $D(B)$ is a B-bimodule with structure given by $(afb)(x) = f(bxa)$ for any $a, b, x \in B$, $f \in D(B)$.

Remark 2.16. Let $C = \alpha_r^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_{r-1}^{m+1}$ be an elementary path with $\overline{\alpha_1 \cdots \alpha_t} = \mu \bar{p}$, $p \in \mathbf{P}$, and $0 \neq \mu \in k$. Then $\phi(C) \neq 0$. In fact, $\phi(C) = \overline{\alpha_r \cdots \alpha_t}^m \bar{p}^* \overline{\alpha_1 \cdots \alpha_{r-1}}^{m+1} \in \widehat{B}(e_{m+1,\, t(\alpha_{r-1})}, e_{m,\, s(\alpha_r)}) \subseteq D(B)$ and

$$\begin{aligned}(\overline{\alpha_r \cdots \alpha_t}\, \bar{p}^* \overline{\alpha_1 \cdots \alpha_{r-1}})(e_{s(\alpha_r)}) &= \bar{p}^*(\overline{\alpha_1 \cdots \alpha_{r-1} \alpha_r \cdots \alpha_t}) \\ &= \bar{p}^*(\mu \bar{p}) = \mu \neq 0.\end{aligned}$$

Remark 2.17. If a path q in $kQ_{\widehat{B}}$ has a supplement q' then $\phi(q) \neq 0$. This is immediate by the previous remark since $\phi(qq') \neq 0$.

Remark 2.18. If γ^{m+1} is a supplement of a path q in $kQ_{\widehat{B}}$ and $\bar{\delta} = \lambda \bar{\gamma}$, $0 \neq \lambda \in k$, then δ^{m+1} is also a supplement of q. In fact, let $q = \alpha_s^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1}$, and let γ^{m+1} be a supplement in the elementary path $C = \alpha_s^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1} \gamma^{m+1}$. Since

$$\mu \bar{p} = \overline{\alpha_1 \cdots \alpha_{s'} \gamma \alpha_s \cdots \alpha_t} = \frac{1}{\lambda} \overline{\alpha_1 \cdots \alpha_{s'} \delta \alpha_s \cdots \alpha_t},$$

then $C' = \alpha_s^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1} \delta^{m+1}$ is also an elementary path, and hence δ^{m+1} is a supplement of q.

PROPOSITION 2.19. *Let $\phi\colon kQ_{\widehat{B}} \to \widehat{B}$ be a surjective morphism of algebras such that $\phi(\alpha^m) = \bar{\alpha}^m$ and $\phi(\beta_{p_i}^m) = \overline{p_i}^*$. Then the ideal $I_\phi = \operatorname{Ker}\phi$ is admissible and is generated by*

(i) *paths $\alpha_s^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_s^{m+1}$ and $\beta_p^{m-1} \alpha_1^m \cdots \alpha_t^m \beta_p^m$ for any elementary path $\alpha_1^m \cdots \alpha_t^m \beta_p^m$;*

(ii) *paths whose arrows do not belong to an elementary path*;

(iii) *elements $q_1 - \mu q_2$ with q_1 and q_2 paths sharing starting and terminal points such that*

(a) *$q_1 = \gamma_1^m$, $q_2 = \gamma_2^m$, $\overline{\gamma_1} = \mu \overline{\gamma_2}$ with $0 \neq \mu \in k$, γ_1, γ_2 paths in Q_B, or*

(b) *q_1 and q_2 have the same supplement γ^{m+1} in elementary paths for a path γ in kQ_B, that is,*

$$\begin{aligned} q_1 \gamma^{m+1} &= \alpha_s^m \cdots \alpha_t^m \beta_{p_1}^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1} \gamma^{m+1}, \\ q_2 \gamma^{m+1} &= \delta_r^m \cdots \delta_l^m \beta_{p_2}^m \delta_1^{m+1} \cdots \delta_{r'}^{m+1} \gamma^{m+1}, \end{aligned}$$

where $p_1, p_2 \in \mathbf{P}$, $s' < s$, and $r' < r$ such that

$$\overline{\alpha_1 \cdots \alpha_{s'} \gamma \alpha_s \cdots \alpha_t} = \lambda_1 \overline{p_1}, \qquad \overline{\delta_1 \cdots \delta_{r'} \gamma \delta_r \cdots \delta_l} = \lambda_2 \overline{p_2},$$

with $0 \neq \mu = \lambda_1 / \lambda_2$.

Proof. It follows from the definition of ϕ that $\mathrm{Ker}\phi$ is admissible. Let I_ϕ be the ideal generated by the relations given in (i), (ii), and (iii). We will first show that $I_\phi \subseteq \ker\phi$.

(i)

$$\phi(\beta_p^{m-1}\alpha_1^m \cdots \alpha_t^m \beta_p^m) = \bar{p}^* \overline{\alpha_1 \cdots \alpha_t}^m \bar{p}^* = 0$$

because $D(B)^2 = 0$, and

$$\phi(\alpha_s^m \cdots \alpha_t^m \beta_p^m \alpha_1^{m+1} \cdots \alpha_s^{m+1}) = \overline{\alpha_s \cdots \alpha_t}^m \bar{p}^* \overline{\alpha_1 \cdots \alpha_s}^{m+1} = 0$$

because $(\overline{\alpha_s \cdots \alpha_t}\,\bar{p}^* \overline{\alpha_1 \cdots \alpha_s})(\bar{x}) = \bar{p}^*(\overline{\alpha_1 \cdots \alpha_s x \alpha_s \cdots \alpha_t}) = 0$ for every $\bar{x} \in B$, since B triangular implies $\alpha_s x \alpha_s = 0$.

(ii) Let q be a path from (m, i) to (s, j) whose arrows do not belong to an elementary path. If $s \neq m, m+1$, then $\phi(q) = 0$.

If $s = m$, then $q = \gamma^m$ for $\gamma = \alpha_1 \cdots \alpha_t$, and so $\phi(q) = \bar{\gamma}^m$. If $\bar{\gamma} \neq 0$, there exists a maximal path $p \in \mathbf{P}$ such that $\overline{\gamma_1 \gamma \gamma_2} = \lambda \bar{p}$ with $0 \neq \lambda \in k$, γ_1, γ_2 paths in kQ_B. This implies that the arrows of q belong to the elementary path $\gamma_1^m \gamma^m \gamma_2^m \beta_p^m$, a contradiction. Then $\phi(q) = \bar{\gamma}^m = 0$.

If $s = m+1$, then we may put $q = \alpha_1^m \cdots \alpha_{r-1}^m \beta_p^m \alpha_{r+1}^{m+1} \cdots \alpha_t^{m+1}$, and hence

$$\phi(q) = \overline{\alpha_1 \cdots \alpha_{r-1}}^m \bar{p}^* \overline{\alpha_{r+1} \cdots \alpha_t}^{m+1}.$$

Now, let x be a path in kQ_B. Then

$$(\overline{\alpha_1 \cdots \alpha_{r-1}}\,\bar{p}^* \overline{\alpha_{r+1} \cdots \alpha_t})(\bar{x}) = \bar{p}^*(\overline{\alpha_{r+1} \cdots \alpha_t x \alpha_1 \cdots \alpha_{r-1}})$$

is nonzero precisely when the path x is such that $\overline{\alpha_{r+1} \cdots \alpha_t x \alpha_1 \cdots \alpha_{r-1}} = \lambda \bar{p}$, $0 \neq \lambda \in k$. But in this case q would have arrows belonging to the elementary path $\alpha_1^m \cdots \alpha_{r-1}^m \beta_p^m \alpha_{r+1}^{m+1} \cdots \alpha_t^{m+1} x^{m+1}$, contradicting our hypothesis. So $\phi(q) = 0$.

(iii) (a) It is immediate since $\phi(q_1 - \mu q_2) = \overline{\gamma_1}^m - \mu \overline{\gamma_2}^m$.

(b) Let

$$q_1 \gamma^{m+1} = \alpha_s^m \cdots \alpha_t^m \beta_{p_1}^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1} \gamma^{m+1},$$

$$q_2 \gamma^{m+1} = \delta_r^m \cdots \delta_l^m \beta_{p_2}^m \delta_1^{m+1} \cdots \delta_{r'}^{m+1} \gamma^{m+1},$$

where $p_1, p_2 \in \mathbf{P}$, $s' < s$, $r' < r$, and γ is a path in kQ_B from i to j such that

$$\overline{\alpha_1 \cdots \alpha_{s'} \gamma \alpha_s \cdots \alpha_t} = \lambda_1 \overline{p_1}, \qquad \overline{\delta_1 \cdots \delta_{r'} \gamma \delta_r \cdots \delta_l} = \lambda_2 \overline{p_2},$$

with $0 \neq \mu = \lambda_1/\lambda_2$. Then

$$\phi(q_1) = \overline{\alpha_s \cdots \alpha_t}^m \overline{p_1}^* \overline{\alpha_1 \cdots \alpha_{s'}}^{m+1}$$

and

$$\phi(q_2) = \overline{\delta_r \cdots \delta_l}^{m}\,\overline{p_2}^{*}\,\overline{\delta_1 \cdots \delta_{r'}}^{m+1}.$$

Let x be an arbitrary path in kQ_B. Then

$$(\overline{\alpha_s \cdots \alpha_t}\,\overline{p_1}^{*}\,\overline{\alpha_1 \cdots \alpha_{s'}})(\bar{x}) = \overline{p_1}^{*}(\overline{\alpha_1 \cdots \alpha_{s'} x \alpha_s \cdots \alpha_t})$$

and

$$(\overline{\delta_r \cdots \delta_l}\,\overline{p_2}^{*}\,\overline{\delta_1 \cdots \delta_{r'}})(\bar{x}) = \overline{p_2}^{*}(\overline{\delta_1 \cdots \delta_{r'} x \delta_r \cdots \delta_l}).$$

If $s(x) \neq s(\gamma)$ or $t(x) \neq t(\gamma)$, the preceding expressions are both equal to zero since $\alpha_{s'} x \alpha_s = 0 = \delta_{r'} x \delta_r$. This is also the case if $\bar{x} = 0$. Assume $\bar{x} \neq 0$ and $s(x) = s(\gamma)$, $t(x) = t(\gamma)$. Since B is schurian, $\bar{x} = \mu\bar{\gamma}$ for some $0 \neq \mu \in k$. So

$$\overline{p_1}^{*}(\overline{\alpha_1 \cdots \alpha_{s'} x \alpha_s \cdots \alpha_t}) = \mu\overline{p_1}^{*}(\lambda_1\overline{p_1}) = \mu\lambda_1$$

and

$$\overline{p_2}^{*}(\overline{\delta_1 \cdots \delta_{r'} x \delta_r \cdots \delta_l}) = \mu\overline{p_2}^{*}(\lambda_2\overline{p_2}) = \mu\lambda_2,$$

hence $\phi(q_1) = (\lambda_1/\lambda_2)\phi(q_2)$, that is, $\phi(q_1 - (\lambda_1/\lambda_2)q_2) = 0$.

To finish the proof we are going to compute $\dim_k kQ_{\widehat{B}}/I_\phi((s,j),(m,i))$. We have $\dim_k kQ_{\widehat{B}}/I_\phi((s,j),(m,i)) = 0$ if $s \neq m, m+1$ by the relations described in (ii).

Consider the case $s = m$. If q is a nonzero path in B the arrows of q belong to an elementary path. So $\dim_k kQ_{\widehat{B}}/I_\phi((m,j),(m,i)) = \dim_k e_i B e_j$, by the relations described in (ii) and (iii)(a).

Finally, let $s = m+1$. By (i), $\dim_k kQ_{\widehat{B}}/I_\phi((m+1,i),(m,i)) = \dim_k e_i D(B) e_i$, and by (ii) and (iii)(b), $\dim_k kQ_{\widehat{B}}/I_\phi((m+1,j),(m,i)) = \dim_k e_i D(B) e_j$ for all i, j with $i \neq j$.

So $\dim_k kQ_{\widehat{B}}/I_\phi((s,j),(m,i)) = \dim_k \widehat{B}(e_{s,j}, e_{m,i})$. Hence $I_\phi = \operatorname{Ker}\phi$. ∎

EXAMPLE 2.20. Let $B = kQ_B$ be the bound quiver algebra given by the quiver

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "\alpha_1"] & 2 \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha_3"'] & \\ & & 4 \end{tikzcd}.$$

So $\mathbf{P} = \{\alpha_1\alpha_2, \alpha_1\alpha_3\}$, and denote $\beta_i = \beta_{\alpha_1\alpha_i}$ for $i = 2, 3$. Then $\widehat{B} = kQ_{\widehat{B}}/I_{\widehat{B}}$ where $Q_{\widehat{B}}$ is the quiver

$$
\begin{tikzcd}
 & & (-1,3) \arrow[dr, "\beta_2^{-1}"] & & & (0,3) \arrow[dr, "\beta_2^{0}"] & & & & \\
\cdots \arrow[r] & (-1,2) \arrow[ur, "\alpha_2^{-1}"] \arrow[dr, "\alpha_3^{-1}"'] & & (0,1) \arrow[r, "\alpha_1^{0}"] & (0,2) \arrow[ur, "\alpha_2^{0}"] \arrow[dr, "\alpha_3^{0}"'] & & (1,1) \arrow[r, "\alpha_1^{1}"] & (1,2) \arrow[ur, "\alpha_2^{1}"] \arrow[dr, "\alpha_3^{1}"'] & & \cdots \\
 & & (-1,4) \arrow[ur, "\beta_3^{-1}"'] & & & (0,4) \arrow[ur, "\beta_3^{0}"'] & & & &
\end{tikzcd}
$$

and the ideal $I_{\widehat{B}}$ is generated by

(i) paths of length 4;

(ii) $\beta_2^m \alpha_1^{m+1} \alpha_3^{m+1}$, $\beta_3^m \alpha_1^{m+1} \alpha_2^{m+1}$;

(iii) $\alpha_2^m \beta_2^m - \alpha_3^m \beta_3^m$.

EXAMPLE 2.21. Let $B = kQ_B/I_B$ be the bound quiver algebra given by the quiver

$$
\begin{tikzcd}
 & 2 \arrow[dr, "\alpha_2"] & \\
1 \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha_3"] & & 3
\end{tikzcd}
$$

and by the ideal I_B in kQ_B generated by $\alpha_1\alpha_2$. So $\mathbf{P} = \{\alpha_1, \alpha_2, \alpha_3\}$, and denote $\beta_i = \beta_{\alpha_i}$ for $i = 1, 2, 3$. Then $\widehat{B} = kQ_{\widehat{B}}/I_{\widehat{B}}$ where $Q_{\widehat{B}}$ is the quiver

$$
\begin{tikzcd}
 & & (0,2) \arrow[dr, "\alpha_2^{0}"] \arrow[drr, "\beta_1^{0}"] & & & (1,2) \arrow[dr, "\alpha_2^{1}"] \arrow[drr, "\beta_1^{1}"] & & & & \\
\cdots \arrow[r, "\beta_3^{-1}"'] & (0,1) \arrow[ur, "\alpha_1^{0}"] \arrow[rr, "\alpha_3^{0}"'] & & (0,3) \arrow[urr, "\beta_2^{0}"] \arrow[r, "\beta_3^{0}"'] & (1,1) \arrow[ur, "\alpha_1^{1}"] \arrow[rr, "\alpha_3^{1}"'] & & (1,3) \arrow[urr, "\beta_2^{1}"] \arrow[r, "\beta_3^{1}"'] & (2,1) \arrow[ur] \arrow[rr] & & \cdots
\end{tikzcd}
$$

and the ideal $I_{\widehat{B}}$ is generated by

(i) paths of length 3;

(ii) $\alpha_1^m \alpha_2^m$, $\alpha_2^m \beta_3^m$, $\alpha_3^m \beta_2^m$, $\beta_1^m \alpha_3^{m+1}$, $\beta_3^m \alpha_1^{m+1}$, $\beta_2^m \beta_1^{m+1}$;

(iii) $\alpha_1^m \beta_1^m - \alpha_3^m \beta_3^m$, $\alpha_2^m \beta_2^m - \beta_1^m \alpha_1^{m+1}$, $\beta_2^m \alpha_2^{m+1} - \beta_3^m \alpha_3^{m+1}$.

3. THE UNIVERSAL GALOIS COVERING

Given a presentation $(Q_{\widehat{B}}, I_{\widehat{B}})$ of $\widehat{B}$, we are going to describe the corresponding presentation (Q_A, I_A) of A, using the Galois covering $F: \widehat{B} \to A$ given in Theorem 2.4 and the induced quiver morphism Q_F. In fact, to describe $\pi_1(Q_A, I_A)$ we only need the minimal relations in I_A. From Remark 2.1 we know that $Q_A = Q_F(Q_{\widehat{B}})$, and it follows from Remark 2.2 that the minimal relations in I_A belong to $Q_F(I_{\widehat{B}})$.

3.1. *Presentations of Self-Injective Algebras $\widehat{B}/G$*

We begin this section with several lemmas which will be useful for describing the ordinary quiver Q_A of A.

Given $i \in (Q_B)_0$, let $\nu^m(e_i) = e_{\nu^m(i)}$. Recall that $\eta(i) = \min\{k > 0 : \nu^k(i) \in (Q_B)_0\}$ and $g(e_{m,i}) = e_{m-\eta(i),\nu^{\eta(i)}(i)}$, as defined after Theorem 2.4.

LEMMA 3.1. *Given $s \in \mathbf{Z}$, $j \in (Q_B)_0$, there exists $l \in \mathbf{Z}$ such that $g^l(e_{s,j}) = e_{m,i}$ with $0 \le m < \eta(i)$, $i \in (Q_B)_0$.*

Proof. It follows from the definition of g that $i \in (Q_B)_0$ if $g^l(e_{s,j}) = e_{m,i}$ for $l \in \mathbf{Z}$. Suppose $s \ge 0$. Define $s_0 = s$, $j_0 = j$, $s_{r+1} = s_r - \eta(j_r)$, and $j_{r+1} = \nu^{\eta(j_r)}(j_r)$. Hence $g(e_{s_r,j_r}) = e_{s_{r+1},j_{r+1}}$. Since $s > s_1 > s_2 > \cdots$ there exists $l \in \mathbf{Z}$ such that $s_l \ge 0$ and $s_{l+1} < 0$. This implies that $0 \le s_l = s_{l+1} + \eta(j_l) < \eta(j_l)$. Hence $g^l(e_{s,j}) = e_{s_l,j_l}$ and $0 \le s_l < \eta(j_l)$.

If $s < 0$, $g^{-s}(e_{s,i}) = e_{s',i'}$ with $s' \ge 0$, by the definition of g. Now we know that there exists $l \in \mathbf{Z}$ such that $g^l(e_{s',i'}) = e_{s'',i''}$ with $0 \le s'' < \eta(i'')$. Hence $g^{l-s}(e_{s,i}) = e_{s'',i''}$, and the proof is complete. ∎

LEMMA 3.2. *For $i, j \in (Q_B)_0$, $\nu^m(i) = \nu^s(j)$ with $0 \le m < \eta(i)$ and $0 \le s < \eta(j)$ imply that $m = s$ and $i = j$.*

Proof. If $m = s$ then $i = j$ since ν is an automorphism. Suppose $m > s$; then $\nu^{m-s}(i) = j \in (Q_B)_0$. Hence $m - s \ge \eta(i)$ by the definition of η. So $m \ge \eta(i)$, a contradiction. ∎

LEMMA 3.3 [22, Lemma 3.4]. *Assume $i, j \in (Q_B)_0$ and $e_jBe_i \ne 0$. Then $\eta(j) = \eta(i)$ or $\eta(j) = \eta(i) + 1$.*

We are now in a position to describe the ordinary quiver Q_A of A, using the quiver morphism $Q_F\colon Q_{\widehat{B}} \to Q_A$ induced by the Galois covering $F\colon \widehat{B} \to A$ and the description of the ordinary quiver $Q_{\widehat{B}}$ given in Proposition 2.13.

PROPOSITION 3.4. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group G as in Theorem 2.4, and assume that B is schurian. Then*

$$(Q_A)_0 = \{\nu^m(i)\}_{i\in(Q_B)_0,\, 0\le m<\eta(i)}$$

and

$$(Q_A)_1 = \{Q_F(\alpha^m)\}_{\alpha\in(Q_B)_1,\, 0\le m<\eta(s(\alpha))} \cup \{Q_F(\beta_p^m)\}_{p\in\mathbf{P},\, 0\le m<\eta(t(p))}.$$

Moreover, Q_A has no double arrows.

Proof. Let $V = \{\nu^m(i)\}_{i\in(Q_B)_0,\, 0\le m<\eta(i)}$. Since $(Q_A)_0 = Q_F((Q_{\widehat{B}})_0)$ and $Q_F(m, i) = \nu^m(i)$, we have $V \subseteq (Q_A)_0$.

To see that $(Q_A)_0 \subseteq V$ it is enough to use that $Fg = F$ and Lemma 3.1, because, given $(s, j) \in (Q_{\widehat{B}})_0$, there exists $l \in \mathbf{Z}$ such that $g^l(e_{s,j}) = e_{m,i}$ with $0 \le m < \eta(i)$, $i \in (Q_B)_0$. Then $F(e_{s,j}) = F(g^l(e_{s,j})) = F(e_{m,i})$. So $\nu^s(j) = \nu^m(i)$.

The elements in V are pairwise different by Lemma 3.2.

Let $S_1 = \{Q_F(\alpha^m)\}_{\alpha\in(Q_B)_1,\, 0\le m<\eta(s(\alpha))}$ and $S_2 = \{Q_F(\beta_p^m)\}_{p\in\mathbf{P},\, 0\le m<\eta(t(p))}$. We know that $S_1 \cup S_2 \subseteq (Q_A)_1$ since $(Q_A)_1 = Q_F((Q_{\widehat{B}})_1)$.

Let $l \in \mathbf{Z}$ be such that $g^l(e_{m,i}) = e_{m',i'}$, $0 \le m' < \eta(i')$. If $F(\widehat{B}(e_{s,j}, e_{m,i})) = F(\widehat{B}(g^l(e_{s,j}), g^l(e_{m,i}))) \ne 0$ then $g^l(e_{s,j}) = e_{s',j'}$ with $s' = m'$ or $s' = m' + 1$. So, for any arrow α: $(m, i) \to (s, j)$, we have $F(\alpha) \in F(\widehat{B}(e_{s',j'}, e_{m',i'}))$ with $s' = m'$ or $s' = m' + 1$ and $0 \le m' < \eta(i')$. Hence $Q_F(\alpha) \in S_1$ if $s' = m'$ and $Q_F(\alpha) \in S_2$ if $s' = m' + 1$. Therefore $(Q_A)_1 \subseteq S_1 \cup S_2$.

To finish the proof we have to show that the arrows are pairwise different. We have several cases to consider:

(a) $Q_F(\alpha^m), Q_F(\gamma^s) \in S_1$, α^m: $(m, i) \to (m, j)$ and γ^s: $(s, i') \to (s, j')$. Now, $Q_F(\alpha^m) = Q_F(\gamma^s)$ implies that $\nu^m(i) = \nu^s(i')$ and $\nu^m(j) = \nu^s(j')$. From Lemma 3.2 we get $m = s$, $i = i'$, and this implies that also $j = j'$. So α, γ are arrows in Q_B sharing starting and terminal points. Since B is schurian Q_B has no double arrows, so $\alpha = \gamma$.

(b) $Q_F(\beta_p^m), Q_F(\beta_{p'}^s) \in S_2$, β_p^m: $(m, i) \to (m+1, j)$ and $\beta_{p'}^s$: $(s, i') \to (s + 1, j')$. Now, $Q_F(\beta_p^m) = Q_F(\beta_{p'}^s)$ implies that $\nu^m(i) = \nu^s(i')$ and $\nu^{m+1}(j) = \nu^{s+1}(j')$. From Lemma 3.2 we get $m = s$, $i = i'$, and this implies that also $j = j'$. Now p, p' are two maximal paths in $\mathbf{P}$ sharing starting and terminal points and B is schurian, so $p = p'$ by the definition of the set $\mathbf{P}$.

(c) $Q_F(\alpha^m) \in S_1$, $Q_F(\beta_p^s) \in S_2$, α^m: $(m, i) \to (m, j)$, and β_p^s : $(s, i') \to (s + 1, j')$. In this case, $Q_F(\alpha^m) = Q_F(\beta_p^s)$ implies that $\nu^m(i) = \nu^s(i')$ and $\nu^m(j) = \nu^{s+1}(j')$. From Lemma 3.2 we get $m = s$, $i = i'$, and this implies that $j = \nu(j')$, hence $\eta(j') = 1$. Now p is a maximal path in kQ_B from j' to i', so $e_{j'}Be_{i'} \ne 0$. From Lemma 3.3 we have $\eta(i') \le \eta(j')$, so $\eta(i') = 1$. But $0 \le s < \eta(i')$, so $s = 0$. Then $\alpha^m = \alpha^0$: $(0, i) \to (0, j)$ and $\beta_p^s = \beta_p^0$: $(0, i) \to (1, j')$. But $Q_F(\alpha^0) \in e_i(\operatorname{rad} B)e_j$ and $Q_F(\beta_p^0) \in e_iIe_{\nu(j')} = e_iIe_j$ by the definition of F, which is impossible [21, Proposition 1.7]. ∎

Now, if $(Q_{\widehat{B}}, I_{\widehat{B}})$ is the presentation of $\widehat{B}$ described in Propositions 2.13 and 2.19, and Q_A is the ordinary quiver of A, we are going to describe the corresponding ideal I_A using the Galois covering F: $\widehat{B} \to A$ given in Theorem 2.4 and the induced quiver morphism Q_F: $Q_{\widehat{B}} \to Q_A$.

From Remark 2.2 we know that the minimal relations in I_A belong to $Q_F(I_{\widehat{B}})$. In fact, for any minimal relation $\rho \in I_A$ there exists $\rho' \in I_{\widehat{B}}$ such that $Q_F(\rho') = \rho$ and $\rho' = \sum \lambda_i w_i$ with $w_i \notin I_{\widehat{B}}$.

So we have to describe $Q_F(\rho')$ for any minimal relation $\rho' \in I_{\widehat{B}}$ described in Proposition 2.19(iii).

Notation. If $q = \alpha_1 \cdots \alpha_r$ is a path in kQ_B, we denote by $q^m = \alpha_1^m \cdots \alpha_r^m$ the corresponding paths in $kQ_{\widehat{B}}$, and, by $\alpha(m) = Q_F(\alpha^m)$, $\beta_p(m) = Q_F(\beta_p^m)$, and $q(m) = Q_F(q^m)$ the corresponding paths in kQ_A.

So we can now describe the relations in I_A.

PROPOSITION 3.5. *Let $(Q_{\widehat{B}}, I_{\widehat{B}})$ be the presentation of $\widehat{B}$ given in Propositions* 2.13 *and* 2.19, *and let Q_F: $Q_{\widehat{B}} \to Q_A$ be the quiver morphism induced by the Galois covering F: $\widehat{B} \to A$ given in Theorem* 2.4. *Then the ideal I_A is generated by*

(i) *paths $\alpha_s(m) \cdots \alpha_t(m)\beta_p(m)\alpha_1(m+1) \cdots \alpha_s(m+1)$ and $\beta_p(m-1)\alpha_1(m) \cdots \alpha_t(m)\beta_p(m)$ for any elementary path $\alpha_1^m \cdots \alpha_t^m \beta_p^m$, $p \in \mathbf{P}$;*

(ii) *paths $Q_F(q)$ for any path q in $kQ_{\widehat{B}}$ whose arrows do not belong to an elementary path;*

(iii) *elements $\gamma_1(m) - \mu\gamma_2(m)$ with $0 \neq \mu \in k$, γ_1, γ_2 paths in kQ_B sharing starting and terminal points such that $\bar{\gamma}_1 \neq 0 \neq \bar{\gamma}_2$ in B;*

(iv) *elements $Q_F(q_1) - \mu Q_F(q_2)$ with $0 \neq \mu \in k$ and q_1, q_2 paths in $kQ_{\widehat{B}}$ having the same supplement in elementary paths in kQ_B, corresponding to the statement* (iii)(b) *in Proposition* 2.19.

In particular, if C and C' are elementary paths in $kQ_{\widehat{B}}$ sharing starting points, then $Q_F(C) - \mu Q_F(C') \in I_A$.

Proof. The proof follows immediately from Proposition 2.19. Observe that in (iii) we use that B is schurian, so $\overline{\gamma_1} = \mu\overline{\gamma_2}$ for some $0 \neq \mu \in k$; and in (iv) we use that if C and C' are elementary paths starting at (m, i) then C and C' have the same supplement $e_{m+1,i}$. ∎

EXAMPLE 3.6. Let $A = kQ_A/I_A$ be the bound quiver algebra with quiver

$$\begin{tikzcd} & 3 \arrow[dr, "\beta_2"] & \\ 2 \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha_3"'] & & 1 \arrow[ll, "\alpha_1"'] \\ & 4 \arrow[ur, "\beta_3"'] & \end{tikzcd},$$

where the ideal I_A in kQ_A is generated by all paths of length 4, $\beta_2\alpha_1\alpha_3$, $\beta_3\alpha_1\alpha_2$, and $\alpha_2\beta_2 - \alpha_3\beta_3$. Then $A = T(B) = \widehat{B}/(\nu_{\widehat{B}})$ is a repetitive covering self-injective algebra, where B is the algebra considered in Example 2.20, $\nu = id$, $\alpha_i = \alpha_i(m)$, and $\beta_i = \beta_i(m)$ for all $m \in \mathbf{Z}$.

EXAMPLE 3.7. Let $A = kQ_A/I_A$ be the bound quiver algebra with quiver

$$\begin{tikzcd}
& 2 \arrow[dd, "\zeta"] \arrow[dr, shift left, "\theta"] & \\
1 \arrow[ur, "\xi"] \arrow[dr, shift left, "\rho"] & & 4 \arrow[ul, shift left, "\delta"'] \arrow[ll, "\gamma"'] \\
& 3 \arrow[ul, shift left, "\omega"] \arrow[ur, "\epsilon"'] &
\end{tikzcd},$$

where the ideal I_A in kQ_A is generated by $\xi\zeta$, $\zeta\epsilon$, $\rho\omega$, $\theta\delta$, $\epsilon\gamma$, $\omega\rho$, $\gamma\xi$, $\delta\theta$, $\xi\theta - \rho\epsilon$, $\zeta\omega - \theta\gamma$, $\omega\xi - \epsilon\delta$, and $\gamma\rho - \delta\zeta$ (see [21]). Then A is a repetitive covering self-injective algebra, where B is the algebra considered in Example 2.21, $\nu(1) = 4$, $\nu(2) = 1$, $\nu(3) = 2$, $\nu(4) = 3$, $\eta(1) = 2$, $\eta(2) = \eta(3) = 1$, $\xi = \alpha_1(0) = \alpha_2(1)$, $\gamma = \alpha_1(1)$, $\zeta = \alpha_2(0) = \beta_3(1)$, $\rho = \alpha_3(0) = \beta_1(1)$, $\delta = \alpha_3(1)$, $\theta = \beta_1(0) = \beta_2(1)$, $\omega = \beta_2(0)$, and $\epsilon = \beta_3(0)$.

3.2. *Fundamental Groups*

In this section we are going to describe the relationship between the fundamental groups of (Q_A, I_A) and (Q_B, I_B).

As we observe after Proposition 2.13, to describe the fundamental groups we may assume without loss of generality that the hypotheses of Proposition 2.19 are satisfied.

Let $i \in (Q_B)_0$, and let $p \in \mathbf{P}$ be a maximal path going through i, that is, $p = p_- p_+$ with $i = s(p_+) = t(p_-)$. For any pair (i, p) where p is a maximal path going through i, we consider the following path in kQ_A: $C_i^m(p) = Q_F(p_+^m \beta_p^m p_-^{m+1})$.

Let $\sim$ be the equivalence relation on the set W of all walks in Q_A defined in Section 2.1.

The element defined by the path $C_i^m(p)$ in the fundamental group of (Q_A, I_A) does not depend on the choice of p, as we prove in the following lemma.

LEMMA 3.8. *For any* $i \in (Q_B)_0$, $C_i^m(p) \sim C_i^m(q)$ *for any* $p, q \in \mathbf{P}$ *going through* i.

Proof. Let $p = p_- p_+$, $q = q_- q_+$, and $i = s(p_+) = t(p_-) = s(q_+) = t(q_-)$. Then $C = p_+^m \beta_p^m p_-^{m+1}$ and $C' = q_+^m \beta_q^m q_-^{m+1}$ are two elementary paths starting at (m, i). So the result follows from Proposition 3.5(iv) because $Q_F(C) - \mu Q_F(C')$ is a minimal relation in I_A. ∎

To simplify the notation, for any $i \in (Q_B)_0$ we fix a maximal path $p \in \mathbf{P}$ going through i and we denote by $C_i^m = C_i^m(p)$ the corresponding path in kQ_A.

In the following lemmas we point out some commutativity relations that will be used in the proof of the main theorem of this section.

LEMMA 3.9. *For any arrow* α: $i \to j$ *in* Q_B *and* m *an integer,*

$$\alpha(m)C_j^m \sim C_i^m \alpha(m+1) \qquad \text{and} \qquad \alpha(m)^{-1}C_i^m \sim C_j^m \alpha(m+1)^{-1}.$$

Proof. Any arrow α: $i \to j$ in Q_B belongs to a maximal path $p_1 \alpha p_2$ in kQ_B. Let $p \in \mathbf{P}$ such that $\bar{p} = \mu \overline{p_1 \alpha p_2}$, $0 \neq \mu \in k$. Then $C_i^m \sim Q_F(\alpha^m p_2^m \beta_p^m p_1^{m+1})$ and $C_j^m \sim Q_F(p_2^m \beta_p^m p_1^{m+1} \alpha^{m+1})$. So $\alpha(m)C_j^m \sim Q_F(\alpha^m p_2^m \beta_p^m p_1^{m+1} \alpha^{m+1}) \sim C_i^m \alpha(m+1)$. ∎

LEMMA 3.10. *For any arrow* α: $i \to j$ *in* Q_B,

(i) *if* $\eta(i) = \eta(j)$ *then*

$$\alpha(\eta(i)-1)C_j^{\eta(i)-1} \sim C_i^{\eta(i)-1}\gamma(0)$$

$$\text{and} \qquad \alpha(\eta(i)-1)^{-1}C_i^{\eta(i)-1} \sim C_j^{\eta(i)-1}\gamma(0)^{-1}$$

for some $\gamma \in (Q_B)_1$;

(ii) *if* $\eta(i) = \eta(j) + 1$ *then*

$$\alpha(\eta(i)-1)C_j^{\eta(i)-1} \sim C_i^{\eta(i)-1} C^0_{\nu^{\eta(i)}(i)} q(1)^{-1}$$

$$\text{and} \qquad \alpha(\eta(i)-1)^{-1}C_i^{\eta(i)-1} \sim q(0)$$

for some $q \in \mathbf{P}$.

Proof. (i) If $\eta(i) = \eta(j)$, then $Q_F(\alpha^{\eta(i)}) = Q_F(\gamma^0) = \gamma(0)$ for some arrow γ: $\nu^{\eta(i)}(i) \to \nu^{\eta(i)}(j)$ in $(Q_B)_1$; it follows from Lemma 3.9 that $\alpha(\eta(i)-1)C_j^{\eta(i)-1} \sim C_i^{\eta(i)-1}\alpha(\eta(i)) = C_i^{\eta(i)-1}\gamma(0)$ and $\alpha(\eta(i)-1)^{-1}C_i^{\eta(i)-1} \sim C_j^{\eta(i)-1}\alpha(\eta(i))^{-1} = C_j^{\eta(i)-1}\gamma(0)^{-1}$.

(ii) If $\eta(i) = \eta(j)+1$, then $Q_F(\alpha^{\eta(i)}) = Q_F(\beta_q^0) = \beta_q(0)$ for some arrow β_q^0: $(0, \nu^{\eta(i)}(i)) \to (1, \nu^{\eta(j)}(j))$ and q: $\nu^{\eta(j)}(j) \to \nu^{\eta(i)}(i)$ a maximal path in $\mathbf{P}$. Since

$$\beta_q(0) \sim \beta_q(0)q(1)q(1)^{-1} \sim C^0_{\nu^{\eta(i)}(i)} q(1)^{-1},$$

the first relation follows from Lemma 3.9. To prove the second one, let $p = p_1 \alpha p_2$ be a maximal path in kQ_B. From Lemma 3.3 we get $\eta(s(p_1)) = \eta(i)$ and $\eta(t(p_2)) = \eta(j)$ since $\eta(s(p_1)) \geq \eta(i) = \eta(j)+1 \geq \eta(t(p_2)) + 1$. Hence $Q_F(p_1^{\eta(i)}) = Q_F(q_1^0) = q_1(0)$, $Q_F(p_2^{\eta(j)}) = Q_F(q_2^0) = q_2(0)$, and $Q_F(\beta_p^{\eta(j)}) = Q_F(\delta^0) = \delta(0)$ for some q_1, q_2 paths in kQ_B, $\delta \in (Q_B)_1$, and $\bar{q} = \psi q_2 \delta q_1$ for some $0 \neq \psi \in k$. So

$$\begin{aligned}\alpha(\eta(i)-1)^{-1}C_i^{\eta(i)-1} &\sim \alpha(\eta(i)-1)^{-1} Q_F(\alpha^{\eta(i)-1} p_2^{\eta(i)-1} \beta_p^{\eta(i)-1} p_1^{\eta(i)}) \\ &\sim Q_F(p_2^{\eta(i)-1} \beta_p^{\eta(i)-1} p_1^{\eta(i)}) = Q_F(q_2^0 \delta^0 q_1^0) = q(0).\end{aligned}$$ ∎

The following theorem shows the relationship between the fundamental groups of (Q_A, I_A) and (Q_B, I_B).

THEOREM 3.11. *Let A be a repetitive covering self-injective algebra, let $F: \widehat{B} \to A$ be a Galois covering with the Galois group G as in Theorem 2.4, and assume that B is schurian. Let (Q_A, I_A) be any presentation of A associated to a presentation (Q_B, I_B) of B. Then we have the following short exact sequence of groups*:

$$1 \to \pi_1(Q_B, I_B) \to \pi_1(Q_A, I_A) \to G \to 1.$$

Proof. We are going to define a surjective morphism of groups $\phi\colon \pi_1(Q_A, I_A) \to G$ whose kernel is $\pi_1(Q_B, I_B)$. Let $w = w_1^{\epsilon_1} \cdots w_n^{\epsilon_n}$ be a walk in Q_A, $\epsilon_i = \pm 1$, $w_i \in (Q_A)_1$, and let $a\colon \pi_1(Q_A, I_A) \to \mathbf{Z}$ be the group morphism given by

$$a([w]) = \sum_{i=1}^{n} (-1)^{\epsilon_i} h(w_i)$$

with

$$h(w_i) = \begin{cases} 1 & \text{if } w_i = \beta_p(0), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $a([w])$ does not depend on a choice of a representative w of the equivalence class $[w]$. In fact, we only have to check the relations described in Proposition 3.5(iii), (iv). If γ_1, γ_2 are paths in kQ_B sharing starting and terminal points, $\bar{\gamma}_1 \neq 0 \neq \bar{\gamma}_2$ in B, then $a([\gamma_1(m)]) = 0 = a([\gamma_2(m)])$. Let q_1, q_2 be paths in $kQ_{\widehat{B}}$ having the same supplement γ^{m+1} in elementary paths, that is,

$$q_1 \gamma^{m+1} = \alpha_s^m \cdots \alpha_t^m \beta_{p_1}^m \alpha_1^{m+1} \cdots \alpha_{s'}^{m+1} \gamma^{m+1},$$
$$q_2 \gamma^{m+1} = \delta_r^m \cdots \delta_l^m \beta_{p_2}^m \delta_1^{m+1} \cdots \delta_{r'}^{m+1} \gamma^{m+1},$$

where $p_1, p_2 \in \mathbf{P}$, $s' < s$, $r' < r$, and γ is a path in kQ_B. Then $q_1(m) \sim \mu q_2(m)$ for some $0 \neq \mu \in k$, and

$$a([q_1(m)]) = \sum_{i=s}^{t} h(\alpha_i(m)) + h(\beta_{p_1}(m)) + \sum_{i=1}^{s'} h(\alpha_i(m+1))$$
$$= \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The same holds for $a([q_2(m)])$, so $a([q_1(m)]) = a([q_2(m)])$. Thus we define $\phi\colon \pi_1(Q_A, I_A) \to G$ by $\phi([w]) = g^{a([w])}$, where g is a fixed generator of G. Clearly ϕ is a group homomorphism.

We will prove next that ϕ is surjective. We fix x_0 as the origin of the paths defining $\pi_1(Q_A, I_A)$ (see Section 2.1). We may assume that x_0 is a source in Q_B. Then there exist a maximal path $p_0 \in \mathbf{P}$ starting at x_0 and a walk u in Q_B from $\nu^{\eta(x_0)}(x_0)$ to x_0. Let $w = p_0(0)\beta_{p_0}(0)p_0(1)\beta_{p_0}(1)\cdots\beta_{p_0}(\eta(x_0)-1)u(0)$. Then $\phi([w]) = g$, so ϕ is surjective.

Next we will show that $\operatorname{Ker}\phi \simeq \pi_1(Q_B, I_B)$. We first observe that

$$\phi([C_x^m]) = \begin{cases} g & \text{if } \ m = 0, \\ 1 & \text{otherwise.} \end{cases}$$

The idea now is to show that any $[w] \in \pi_1(Q_A, I_A)$ can be written in the form

$$[w] = [C_{x_0}^0 C_{x_0}^1 \cdots C_{x_0}^{\eta(x_0)-1} C_{x_1}^0 C_{x_1}^1 \cdots C_{x_1}^{\eta(x_1)-1} C_{x_2}^0 \cdots C_{x_r}^{\eta(x_r)-1} u(0)],$$

where u is a walk in Q_B from $\nu^{\eta(x_r)}(x_r)$ to x_0 and $x_l = \nu^{\eta(x_{l-1})}(x_{l-1})$ for all $l = 1, \cdots, r$. If this is the case, $\phi([w]) = g^{r+1}$, and therefore $[w] \in \operatorname{Ker}\phi$ if and only if $[w] = [u(0)]$ for some walk u in Q_B, as desired.

So we have to prove that any $[w] \in \pi_1(Q_A, I_A)$ can be written as claimed. Clearly either w is $u(0)$ for some walk u in Q_B or w is a composition of walks $u(m)\beta_p(m)v(m+1)$ and their inverses with $p \in \mathbf{P}$ and walks u, v in Q_B. In the first case there is nothing to prove. In the second case we may assume that the walks that occur in w are all of the form $u(m)\beta_p(m)v(m+1)$, or all of the form $u(m+1)\beta_p^{-1}(m)v(m)$, as we prove next. We first observe that $\beta_p(m) \sim p^{-1}(m)p(m)\beta_p(m) \sim p^{-1}(m)C_{s(p)}^m$, and, by iterated use of Lemma 3.9, we have $C_{s(u)}^m u(m+1) \sim u(m)C_{t(u)}^m$ for any walk u in Q_B. Then

$$\begin{aligned} \beta_p(m)v(m+1)\beta_q(m)^{-1} &\sim p^{-1}(m)C_{s(p)}^m v(m+1)(C_{s(q)}^m)^{-1}q(m) \\ &\sim p^{-1}(m)C_{s(p)}^m (C_{s(v)}^m)^{-1} v(m)q(m) \sim (p^{-1}vq)(m). \end{aligned}$$

The last relation holds because $C_{s(p)}^m$ and $C_{s(v)}^m$ correspond to elementary paths sharing starting points since $s(p) = s(v)$ (see Lemma 3.8).

To finish the proof we may assume that w is a composition of walks of the form $u(m)\beta_p(m)v(m+1)$. The case where β_p^{-1} occurs is analogous. For any walk u in Q_B with origin x, $u(m)\beta_p(m) \sim u(m)p^{-1}(m)C_{s(p)}^m$. From Proposition 3.4 we may assume that $0 \le m < \eta(x) - 1$. Now, for any arrow $\alpha\colon i \to j$ appearing in u or p we apply the commutativity relations proved in Lemmas 3.9 and 3.10, depending on m, $\eta(i)$, and $\eta(j)$. So we get

$$u(m)p^{-1}(m)C_{s(p)}^m \sim \begin{cases} C_x^m z(m+1) & \text{with } \ s(z) = x, \\ C_x^{\eta(x)-1} C_{\nu^{\eta(x)}(x)}^0 z(0) & \text{with } \ s(z) = \nu^{\eta(x)}(x), \ \text{ or} \\ z(0) & \text{with } \ s(z) = x \end{cases}$$

for some walk z in Q_B. Iterating this procedure, we get the desired equality,

$$[w] = [C_{x_0}^0 C_{x_0}^1 \cdots C_{x_0}^{\eta(x_0)-1} C_{x_1}^0 C_{x_1}^1 \cdots C_{x_1}^{\eta(x_1)-1} C_{x_2}^0 \cdots C_{x_r}^{\eta(x_r)-1} u(0)],$$

where u is a walk in Q_B from $\nu^{\eta(x_r)}(x_r)$ to x_0, and $x_l = \nu^{\eta(x_{l-1})}(x_{l-1})$ for all $l = 1, \ldots, r$. ∎

Recall that a connected triangular algebra B is called *simply connected* if the fundamental group $\pi_1(Q_B, I_B)$ is trivial for any presentation (Q_B, I_B) of B.

An important consequence of the previous theorem is the following.

COROLLARY 3.12. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group G as in Theorem* 2.4, *and assume that B is schurian. Then $F\colon \widehat{B} \to A$ is the universal Galois covering of A if and only if B is simply connected.*

Proof. The Galois covering $F\colon \widehat{B} \to A$ is universal if and only if $G \simeq \pi_1(Q_A, I_A)$ for any presentation (Q_A, I_A) of A, and hence it follows from Theorem 3.11 that F is universal if and only if $\pi_1(Q_B, I_B)$ is trivial for any presentation (Q_B, I_B) of B. ∎

COROLLARY 3.13. *Let B be a triangular schurian algebra. Then the Galois covering $\widehat{B} \to B \ltimes D(B)$ with the Galois group generated by $\nu_{\widehat{B}}$ is universal if and only if B is simply connected.*

Proof. The trivial extension $T(B) = B \ltimes D(B)$ of B by $D(B)$ is a repetitive covering self-injective algebra, as we have seen in Example 2.6. Hence we have that $\widehat{B} \to T(B) = B \ltimes D(B)$ is the universal Galois covering of $T(B)$ if and only if B is simply connected.

COROLLARY 3.14. *Let $A = kQ/J^{t+1}$, where Q is the oriented cycle with n vertices and n arrows, $1 \leq t \leq n$, and J is the ideal generated by the arrows. Let $B = kQ_B$ be the hereditary algebra with*

$$Q_B\colon 1 \to 2 \to \cdots \to t.$$

Then $\widehat{B} \to A$ is the universal Galois covering with the Galois group $G = (\gamma^n)$ (see Example 2.7).

Proof. It follows from the fact that A is a repetitive covering self-injective algebra and B is simply connected.

COROLLARY 3.15. *Let B be a triangular schurian algebra, $A = \widehat{B}/(\nu_{\widehat{B}}^c)$, $c \neq 0$. Then $\widehat{B} \to A$ is the universal Galois covering with the Galois group $(\nu_{\widehat{B}}^c)$ if and only if B is simply connected.*

Remark 3.16. The representatives of the derived equivalence classes of standard representation–finite self-injective algebras $\Lambda(A_n, s/n, 1)$, $s \geq n$, $\Lambda(A_{2p+1}, s, 2)$, $\Lambda(D_n, s, 1)$, $\Lambda(D_n, s, 2)$, $\Lambda(D_4, s, 3)$, $\Lambda(E_n, s, 1)$, and $\Lambda(E_6, s, 2)$ considered in Example 2.8 are repetitive covering self-injective algebras with $B = kQ_B$, and Q_B is a Dynkin graph (see [1, 2.5]). So Theorem 3.11 shows that their universal Galois coverings are given by $\widehat{B}$.

EXAMPLE 3.17. Let $A = kQ_A/I_A$ be the bound quiver algebra considered in Example 3.6. Since B is simply connected then $\widehat{B}$ is the universal Galois covering of A.

EXAMPLE 3.18. Let $A = kQ_A/I_A$ be the bound quiver algebra considered in Example 3.7. Then $\widehat{B}$ is not the universal Galois covering of A since B is not simply connected.

4. THE FIRST HOCHSCHILD COHOMOLOGY GROUP

Now we want to show the connection between the first Hochschild cohomology group of A and the fundamental group of any presentation (Q_B, I_B) of B. Recall that $H^1(A) \simeq \mathrm{Der}(A)/\mathrm{Inn}(A)$, where $\mathrm{Der}(A) = \{\delta \in \mathrm{Hom}_k(A, A) : \delta(xy) = x\delta(y) + \delta(x)y\}$ is the k-vector space of derivations of A on A, and $\mathrm{Inn}(A) = \{\delta_x\colon A \to A : \delta_x(y) = yx - xy, x \in A\}$ is the subspace of inner derivations.

Let A be a repetitive covering self-injective algebra, $B = A/I$ triangular, and let e be a residual identity of B. Then it is known that

$$0 \to eIe \to eAe \xrightarrow{\rho} eAe/eIe \to 0$$

is a trivial extension of eAe/eIe by eIe; that is, there exists a morphism of algebras $\iota\colon eAe/eIe \to eAe$ such that $\rho\iota = id$ (see [22]). Since eAe/eIe and B are isomorphic A-algebras, eAe is isomorphic to the algebra $B \oplus eIe$ with product $(b, eye)(b', ey'e) = (bb', bey'e + eyeb')$, and from now on we shall identify them. Hence any element in A can be written as

$$a = eae + ea(1-e) + (1-e)a = b + eye + ea(1-e) + (1-e)a,$$

with $b \in B$ and $y \in I$.

THEOREM 4.1. *If A is a repetitive covering self-injective algebra then $H^1(A) \neq 0$.*

Proof. Consider the k-linear map $\delta\colon A \to A$ defined by

$$\delta(a) = \delta(b + eye + ea(1-e) + (1-e)a) = eye + ea(1-e).$$

Recall that $IeI = 0$ and $1 - e \in I$, and observe that $\delta((1-e)A) = 0$ and $\delta(x) = x$ for any $x \in eA(1-e)$ or $x \in eIe$ by the definition of δ. A direct computation shows that δ is a derivation. So it is enough to show that δ is not inner. Suppose contrarily that it is inner, that is, $\delta = \delta_x$ for some $x \in A$. Then, since $\delta(e_i) = 0$ for all $i \in (Q_A)_0$, $0 = \delta(e_i) = \delta_x(e_i) = e_i x - x e_i$, and so $x = \sum_i e_i x = \sum_i e_i x e_i = \sum_i \lambda_i e_i + y$ with $y \in \oplus_i(e_i \operatorname{rad} A\ e_i)$. For any arrow α in Q_B, we have

$$0 = \delta_x(\alpha) = \alpha x - x\alpha = (\lambda_{t(\alpha)} - \lambda_{s(\alpha)})\alpha + \alpha y - y\alpha.$$

Since $\alpha y - y\alpha \in \operatorname{rad}^2 A$ and $\alpha \notin \operatorname{rad}^2 A$, we get $\lambda_{t(\alpha)} = \lambda_{s(\alpha)}$. Since this holds for any arrow α in Q_B and Q_B is connected, we have $\lambda_i = \lambda_j$ for all $i, j \in (Q_B)_0$.

Let i be a fixed source in Q_B, and let $p \in \mathbf{P}$ be a maximal path in kQ_B such that $s(p) = i$. For $\gamma := p\beta_p$ and $0 \le m < \eta(i)$ we have $\gamma(m) = p(m)\beta_p(m)$: $e_{\nu^m(i)} \to e_{\nu^{m+1}(i)}$ are maximal paths in kQ_A (see Proposition 3.5(i), (ii)). By definition, $\delta(\bar{\gamma}(0)) = \bar{\gamma}(0)$ since $\bar{\gamma}(0)$ belongs to eIe or $eA(1-e)$, and $\delta(\bar{\gamma}(m)) = 0$ for $0 < m < \eta(i)$ since $\delta((1-e)A) = 0$. But

$$\begin{aligned}\delta(\bar{\gamma}(m)) &= \delta_x(\bar{\gamma}(m)) = (\lambda_{\nu^{m+1}(i)} - \lambda_{\nu^m(i)})\bar{\gamma}(m) + \bar{\gamma}(m)y - y\bar{\gamma}(m)\\ &= (\lambda_{\nu^{m+1}(i)} - \lambda_{\nu^m(i)})\bar{\gamma}(m)\end{aligned}$$

since $\gamma(m)$ is a maximal path. Then $\lambda_{\nu^m(i)} = \lambda_{\nu^{m+1}(i)}$ for any m with $0 < m < \eta(i)$. But $i, \nu^{\eta(i)}(i)$ are both vertices in Q_B; hence $\lambda_{\nu^{\eta(i)}(i)} = \lambda_i$. So $\lambda_{\nu^m(i)} = \lambda_{\nu^{m+1}(i)}$ for any m with $0 \le m < \eta(i)$ and $\delta(\bar{\gamma}(0)) = \delta_x(\bar{\gamma}(0)) = (\lambda_{\nu(i)} - \lambda_i)\bar{\gamma}(0) = 0$, a contradiction. ∎

Remark 4.2. If A is a repetitive covering self-injective algebra, $A \simeq \widehat{B}/G$ and $B = k$, then $A = kQ/J^2$, where Q is the oriented cycle

$$1 \to \nu(1) \to \nu^2(1) \to \cdots \to \nu^{\eta(1)-1}(1) \to 1,$$

and J is the ideal generated by the arrows. In this case $H^1(A)$ is well known: $H^1(A) = k^2$ if $\nu(1) = 1$ and k is a field of characteristic 2, and $H^1(A) = k$ otherwise [6, Propositions 2.3, 2.4].

From now on we assume that B is not the trivial k-algebra k. This means that the set of vertices Q_0 has at least two elements, since Q has no oriented cycles.

LEMMA 4.3 [14]. *Let A be a basic, connected finite-dimensional algebra, and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents, $n > 1$. For any $\delta \in \operatorname{Der}(A)$ there exists an inner derivation δ_x such that*

$$(\delta - \delta_x)(e_i) = 0 \qquad \textit{for all } i = 1, \dots, n.$$

Proof. Let $\delta: A \to A$ be a derivation. Then

$$\begin{aligned}\delta(e_i) &= \delta(e_ie_i) = e_i\delta(e_i) + \delta(e_i)e_i\\ &= e_i(e_i\delta(e_i) + \delta(e_i)e_i) + (e_i\delta(e_i) + \delta(e_i)e_i)e_i\\ &= e_i\delta(e_i) + \delta(e_i)e_i + 2e_i\delta(e_i)e_i \qquad (1)\end{aligned}$$

For any pair (i, j) with $i \neq j$, we have

$$0 = \delta(e_ie_j) = e_i\delta(e_j) + \delta(e_i)e_j. \qquad (2)$$

Let $x = \sum_{j=1}^{n} e_j\delta(e_j)$. Then $\delta_x(e_i) = e_ix - xe_i = e_i\delta(e_i) - \sum_{j=1,\, j\neq i}^{n} e_j \delta(e_j)e_i - e_i\delta(e_i)e_i$. By (2), we have $\delta_x(e_i) = e_i\delta(e_i) + \sum_{j=1, j\neq i}^{n} e_je_j\delta(e_i) - e_i\delta(e_i)e_i = \delta(e_i) - e_i\delta(e_i)e_i$.

Hence it suffices to show that $e_i\delta(e_i)e_i = 0$, which follows from (1). In fact, (1) implies $2e_i\delta(e_i)e_i = 0$, and so $e_i\delta(e_i)e_i = 0$ if $\operatorname{char} k \neq 2$. In the case where $\operatorname{char} k = 2$, (1) shows that $\delta(e_i) = e_i\delta(e_i) + \delta(e_i)e_i$; hence $e_i\delta(e_i)e_i = 0$ by multiplying e_i from the left. ∎

We denote by $\operatorname{Der}^n(A) = \{\delta \in \operatorname{Der}(A) : \delta(e_i) = 0 \text{ for all } i\}$ the subspace of normalized derivations and $\operatorname{Inn}^n(A) = \operatorname{Der}^n(A) \cap \operatorname{Inn}(A)$.

COROLLARY 4.4.

$$H^1(A) \simeq \operatorname{Der}^n(A)/\operatorname{Inn}^n(A)$$

and $\operatorname{Inn}^n(A) = \{\delta_x :\ x = \sum_{i=1}^{n} \lambda_ie_i + y, y \in \oplus_i e_i \operatorname{rad} Ae_i\}$.

Proof. By the previous lemma we have $H^1(A) \simeq \operatorname{Der}^n(A)/\operatorname{Inn}^n(A)$. Let $\delta_x \in \operatorname{Inn}^n(A)$ for $x \in A$. Then $0 = \delta_x(e_i) = e_ix - xe_i$ for all i. So $x = \sum_{i=1}^{n} e_ix = \sum_{i=1}^{n} e_ixe_i = \sum_{i=1}^{n} \lambda_ie_i + y$ for some $y \in \oplus_i e_i \operatorname{rad} A\ e_i$. ∎

The proof of the following proposition is a special case of [21, Proposition 1.7].

PROPOSITION 4.5. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu_{\widehat{B}}^{c})$, $c \neq 0$, as in Theorem 2.4, and assume that B is schurian. Then, for any arrow $\alpha\colon i \to j$ in Q_A, $\dim_k e_iAe_j = 1$.*

COROLLARY 4.6. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu_{\widehat{B}}^{c})$, $c \neq 0$, as in Theorem 2.4, and assume that B is schurian. If $\delta \in \operatorname{Der}^n(A)$ then $\delta(\alpha) = \lambda_\alpha\alpha$ for any arrow $\alpha \in (Q_A)_1$ and some $\lambda_\alpha \in k$.*

Proof. Let $\delta \in \operatorname{Der}^n(A)$. Since δ is a derivation, for any arrow $\alpha \in (Q_A)_1$ we have $\delta(\alpha) = \delta(e_{s(\alpha)}\alpha e_{t(\alpha)}) = e_{s(\alpha)}\delta(\alpha)e_{t(\alpha)}$. By the previous proposition $\dim_k e_{s(\alpha)}Ae_{t(\alpha)} = 1$, so $\delta(\alpha) = \lambda_\alpha\alpha$. ∎

Remark 4.7. Given $\delta \in \operatorname{Der}^n(A)$ and a path $\gamma = \alpha_1 \cdots \alpha_r$ with α_i arrows in Q_A,

$$\delta(\bar{\gamma}) = \sum_{i=1}^{r} \overline{\alpha_1 \cdots \alpha_{i-1}}\delta(\alpha_i)\overline{\alpha_{i+1} \cdots \alpha_r} = \left(\sum_{i=1}^{r} \lambda_{\alpha_i}\right)\bar{\gamma}.$$

Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu_{\widehat{B}}^{c})$, $c \neq 0$, as in Theorem 2.4, and assume that B is schurian. Let p_0 be a fixed maximal path in kQ_B from i to j. We define a k-linear map L: $\mathrm{Der}^n(A) \to k$ as

$$L(\delta) = \sum_{m=0}^{c-1} \lambda_m,$$

where $\delta(\overline{p_0\beta_{p_0}}(m)) = \lambda_m \overline{p_0\beta_{p_0}}(m)$.

THEOREM 4.8. *The map L induces a surjective group morphism*

$$H^1(A) \xrightarrow{\widehat{L}} k \to 0.$$

Proof. First we will show that $L(\mathrm{Inn}(A)) = 0$. Let δ_x be an inner derivation with $x = \sum \mu_i e_i + y$, $y \in \mathrm{rad}\, A$. Since $\overline{p_0\beta_{p_0}}(m)$ are maximal paths, then

$$\delta_x(\overline{p_0\beta_{p_0}}(m)) = \overline{p_0\beta_{p_0}}(m)x - x\overline{p_0\beta_{p_0}}(m) = (\mu_{\nu^{m+1}(i)} - \mu_{\nu^m(i)})\overline{p_0\beta_{p_0}}(m).$$

So $\lambda_m = \mu_{\nu^{m+1}(i)} - \mu_{\nu^m(i)}$, and $L(\delta_x) = \sum_{m=0}^{c-1} \lambda_m = \mu_{\nu^c(i)} - \mu_i = 0$ since $\nu^c(i) = i$. Hence L induces a morphism $\widehat{L}$: $H^1(A) \to k$.

Let δ be the derivation defined in Theorem 4.1. Then $\delta(\overline{p_0\beta_{p_0}}(m)) = 0$ for any m with $0 < m < c$, because $\delta((1-e)A) = 0$. Moreover, $\delta(\overline{p_0\beta_{p_0}}(0)) = \overline{p_0\beta_{p_0}}(0)$, because, if $c > 1$ or $c = 1$, $\overline{p_0\beta_{p_0}}(0) \in eA(1-e)$ or $\overline{p_0\beta_{p_0}}(0) \in eIe$, respectively. Then $\widehat{L}([\delta]) = 1$. So $\widehat{L}$ is surjective. ∎

Now we will prove several lemmas that will be used in the proof of the main theorem of this section.

Recall that if p is a maximal path in kQ_B that goes through the vertex i in $(Q_B)_0$, we write $p = p_- p_+$ for some subpaths p_-, p_+ with $t(p_-) = i = s(p_+)$.

Let $\delta \in \mathrm{Der}^n(A)$, $i \in (Q_B)_0$, and let p be a maximal path going through i. Denote by $\lambda_p^{(i,m)}$ the element in k such that

$$\delta(\overline{p_+\beta_p}(m)\overline{p_-}(m+1)) = \lambda_p^{(i,m)}\overline{p_+\beta_p}(m)\overline{p_-}(m+1).$$

LEMMA 4.9. *If p, q are maximal paths going through i, then $\lambda_p^{(i,m)} = \lambda_q^{(i,m)}$ for any m with $0 \le m < c$.*

Proof. By Proposition 3.5(iv) we have

$$\overline{p_+\beta_p}(m)\overline{p_-}(m+1) = \mu\overline{q_+\beta_q}(m)\overline{q_-}(m+1)$$

for some $0 \neq \mu \in k$. So

$$\delta(\overline{p_+\beta_p}(m)\overline{p_-}(m+1)) = \mu\delta(\overline{q_+\beta_q}(m)\overline{q_-}(m+1));$$

that is,

$$\lambda_p^{(i,m)}\overline{p_+\beta_p}(m)\overline{p_-}(m+1) = \lambda_q^{(i,m)}\mu\overline{q_+\beta_q}(m)\overline{q_-}(m+1).$$

Since $\overline{p_+\beta_p}(m)\overline{p_-}(m+1) \neq 0$ (see Remark 2.16) we have $\lambda_p^{(i,m)} = \lambda_q^{(i,m)}$. ∎

So we may denote $\lambda^{(i,m)} = \lambda_p^{(i,m)}$ for any maximal path p going through i.

Remark 4.10. If L is the map defined in Theorem 4.8, then

$$\widehat{L}([\delta]) = \sum_{m=0}^{c-1} \lambda_m = \sum_{m=0}^{c-1} \lambda_{p_0}^{(s(p_0),m)} = \sum_{m=0}^{c-1} \lambda^{(s(p_0),m)}.$$

LEMMA 4.11. *For any* $i, j \in (Q_B)_0$, $\sum_{m=0}^{c-1} \lambda^{(i,m)} = \sum_{m=0}^{c-1} \lambda^{(j,m)}$.

Proof. Since Q_B is connected it is enough to show that the assertion holds whenever there exists an arrow $\alpha: i \to j$. Let p be a maximal path containing the arrow α, that is, $p = p_1 \alpha p_2$. Now we claim that $\lambda_{\alpha(0)} = \lambda_{\alpha(c)}$. In fact, $\alpha(c) \in e_{\nu^c(i)} A e_{\nu^c(j)} = e_i A e_j$ and $\dim_k e_i A e_j = \dim_k e_i B e_j = 1$ by Proposition 4.5. Then $\alpha(c) = \mu \alpha(0)$ for some $0 \neq \mu \in k$, and $\delta(\alpha(c)) = \mu \delta(\alpha(0)) = \lambda_{\alpha(0)} \alpha(c)$, while $\delta(\alpha(c)) = \lambda_{\alpha(c)} \alpha(c)$. Therefore it follows that

$$\begin{aligned}
\sum_{m=0}^{c-1} \lambda^{(i,m)} &= \sum_{m=0}^{c-1} \lambda_p^{(i,m)} = \sum_{m=0}^{c-1} (\lambda_{\alpha(m)} + \lambda_{p_2(m)} + \lambda_{p_1(m+1)}) \\
&= \lambda_{\alpha(0)} + \sum_{m=0}^{c-1} (\lambda_{p_2(m)} + \lambda_{p_1(m+1)} + \lambda_{\alpha(m+1)}) - \lambda_{\alpha(c)} \\
&= \sum_{m=0}^{c-1} (\lambda_{p_2(m)} + \lambda_{p_1(m+1)} + \lambda_{\alpha(m+1)}) \\
&= \sum_{m=0}^{c-1} \lambda_p^{(j,m)} = \sum_{m=0}^{c-1} \lambda^{(j,m)}.
\end{aligned}$$

∎

Let (Q_B, I) be a presentation of B and recall the following construction from [18, 1.2]. Let k^+ be the underlying additive group of the field k, and let $C^0(B, I, k^+)$ be the set of all k^+-valued functions on $(Q_B)_0$. Let $Z^1(B, I, k^+)$ be the set of all k^+-valued functions f on $(Q_B)_1$ such that $\sum_{i=1}^{s} f(\alpha_i) = \sum_{j=1}^{t} f(\beta_j)$ whenever there exists a minimal relation $\rho = \sum_{i=1}^{m} \lambda_i w_i$ such that $w_1 = \alpha_1 \alpha_2 \cdots \alpha_s$ and $w_2 = \beta_1 \beta_2 \cdots \beta_t$.

We have an exact sequence of abelian groups,

$$0 \to k^+ \xrightarrow{d_0} C^0(B, I, k^+) \xrightarrow{d_1} Z^1(B, I, k^+) \xrightarrow{p} \mathrm{Hom}(\pi_1(Q_B, I), k^+) \to 0,$$

where $d_0(m): (Q_B)_0 \to k^+$ is the constant function with value m; the map $d_1(g): (Q_B)_1 \to k^+$ is defined by $d_1(g)(\alpha) = g(t(\alpha)) - g(s(\alpha))$, and $p(f) \in \mathrm{Hom}(\pi_1(Q_B, I), k^+)$ is defined by $p(f)([\alpha_1^{\epsilon_1} \cdots \alpha_t^{\epsilon_t}]) = \sum_{i=1}^{t} \epsilon_i f(\alpha_i)$.

THEOREM 4.12. *Let A be a repetitive covering self-injective algebra, let $F\colon \widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu^c_{\widehat{B}})$, $c \neq 0$, as in Theorem* 2.4, *and assume that B is schurian. Then we have the following exact sequence of abelian groups*:

$$0 \to k^+ \xrightarrow{d_0} C^0(B, I, k^+) \xrightarrow{d_1} Z^1(B, I, k^+) \xrightarrow{h} H^1(A) \xrightarrow{\widehat{L}} k \to 0.$$

Proof. We first define the group morphism $h\colon Z^1(B, I, k^+) \to H^1(A)$. Let Q_A be the ordinary quiver of A whose description was given in Proposition 3.4. Let $f \in Z^1(B, I, k^+)$, and extend f to paths by writing $f(\alpha_1 \cdots \alpha_m) := \sum_{i=1}^m f(\alpha_i)$ for $\alpha_i \in (Q_B)_1$. We define a k-linear map $\delta_f\colon A \to A$ as

$$\begin{aligned} \delta_f(e_i) &= 0 && \text{for } i \in (Q_A)_0, \\ \delta_f(\alpha(m)) &= f(\alpha)\alpha(m) && \text{for } \alpha(m) \in (Q_A)_1, 0 \leq m < c, \\ \delta_f(\beta_p(m)) &= -f(p)\beta_p(m) && \text{for } \beta_p(m) \in (Q_A)_1, 0 \leq m < c, \end{aligned}$$

and, for any path $\gamma = \gamma_1 \cdots \gamma_s$, $\gamma_i \in (Q_A)_1$,

$$\delta_f(\bar{\gamma}) = \sum_{i=1}^{s} \overline{\gamma_1 \cdots \gamma_{i-1}} \delta_f(\gamma_i) \overline{\gamma_{i+1} \cdots \gamma_s}.$$

To see that δ_f is well defined we have to check the minimal relations described in Proposition 3.5. If $\overline{\gamma_1} = \mu \overline{\gamma_2}$ in B with $0 \neq \mu \in k$, then $f(\gamma_1) = f(\gamma_2)$ by the definition of $Z^1(B, I, k^+)$. So

$$\delta_f(\overline{\gamma_1}(m)) = f(\gamma_1)\overline{\gamma_1}(m) = f(\gamma_2)\overline{\gamma_1}(m) = f(\gamma_2)\mu\overline{\gamma_2}(m) = \mu\delta_f(\overline{\gamma_2}(m)).$$

If q_1 and q_2 are paths in $kQ_{\widehat{B}}$ having the same supplement $\gamma(m+1)$, say

$$\begin{aligned} q_1\gamma(m+1) &= (\alpha_s \cdots \alpha_t \beta_{p_1})(m)(\alpha_1 \cdots \alpha_{s'}\gamma)(m+1), \\ q_2\gamma(m+1) &= (\delta_r \cdots \delta_l \beta_{p_2})(m)(\delta_1 \cdots \delta_{r'}\gamma)(m+1), \end{aligned}$$

where $p_1, p_2 \in \mathbf{P}$, γ is a path in kQ_B, $s' < s$, $r' < r$, and $0 \leq m < c-1$, then

$$\delta_f(\overline{q_1}) = \left(\sum_{i=s}^{t} f(\alpha_i) - f(p_1) + \sum_{i=1}^{s'} f(\alpha_i) \right) \overline{q_1},$$

$$\delta_f(\overline{q_2}) = \left(\sum_{i=r}^{l} f(\delta_i) - f(p_2) + \sum_{i=1}^{r'} f(\delta_i) \right) \overline{q_2}.$$

Observe that the last equalities are also true for $m + 1 = c$. In fact, given $\alpha\colon i \to j$ in $(Q_B)_1$, $\alpha(c) \in e_{\nu^c(i)} A e_{\nu^c(j)} = e_i A e_j$ and $\dim_k e_i A e_j =$

$\dim_k e_iBe_j = 1$, by Proposition 4.5. Then $\alpha(c) = \mu\alpha(0)$, $0 \neq \mu \in k$, and $\delta_f(\alpha(c)) = \mu\delta_f(\alpha(0)) = \mu f(\alpha)\alpha(0) = f(\alpha)\alpha(c)$.

Now,

$$\overline{\alpha_1 \cdots \alpha_{s'}\gamma\alpha_s \cdots \alpha_t} = \lambda_1 \overline{p_1}, \qquad \overline{\delta_1 \cdots \delta_{r'}\gamma\delta_r \cdots \delta_l} = \lambda_2 \overline{p_2}$$

imply that

$$\sum_{i=1}^{s'} f(\alpha_i) + \sum_{i=s}^{t} f(\alpha_i) - f(p_1) = -f(\gamma) = \sum_{i=1}^{r'} f(\delta_i) + \sum_{i=r}^{l} f(\delta_i) - f(p_2).$$

So $\overline{q_1} = \mu\overline{q_2}$ implies that $\delta_f(\overline{q_1}) = -f(\gamma)\overline{q_1} = -f(\gamma)\mu\overline{q_2} = \mu\delta_f(\overline{q_2})$.

Clearly δ_f is a derivation. Now we define h: $Z^1(B, I, k^+) \to H^1(A)$ as follows: $h(f) = [\delta_f] \in H^1(A)$. To finish the proof we only have to show that the sequence

$$C^0(B, I, k^+) \overset{d_1}{\to} Z^1(B, I, k^+) \overset{h}{\to} H^1(A) \overset{\widehat{L}}{\to} k$$

is exact because the exactness in the other places follows from [18, 1.2] and Theorem 4.8.

(i) $h \cdot d_1 = 0$ since $h(d_1(g)) = [\delta_{d_1(g)}]$ for $g \in C^0(B, I, k^+)$, and we will show that $\delta_{d_1(g)} = \delta_x$ with $x = \sum_{i\in(Q_B)_0}\sum_{m=0}^{c-1} g(i)e_{\nu^m(i)}$. Since we are working with derivations it is enough to show that $\delta_{d_1(g)}(\alpha) = \delta_x(\alpha)$ for any $\alpha \in (Q_A)_1$. Now, for any m with $0 \leq m < c$,

$$\begin{aligned}\delta_{d_1(g)}(\alpha(m)) &= d_1(g)(\alpha)\alpha(m)\\ &= (g(t(\alpha)) - g(s(\alpha))\alpha(m)\\ &= \alpha(m)x - x\alpha(m) = \delta_x(\alpha(m)),\end{aligned}$$

since $s(\alpha(m)) = \nu^m(s(\alpha))$ and $t(\alpha(m)) = \nu^m(t(\alpha))$. For any m with $0 \leq m < c$,

$$\begin{aligned}\delta_{d_1(g)}(\beta_p(m)) &= -d_1(g)(p)\beta_p(m)\\ &= -(g(t(p)) - g(s(p)))\beta_p(m)\\ &= \beta_p(m)x - x\beta_p(m) = \delta_x(\beta_p(m)),\end{aligned}$$

since $s(\beta_p(m)) = \nu^m(t(p))$, $t(\beta_p(m)) = \nu^{m+1}(s(p))$, and $t(\beta_p(c-1)) = \nu^c(s(p)) = s(p)$. Hence $\delta_{d_1(g)} \in \operatorname{Inn}(A)$.

(ii) Let $f \in Z^1(B, I, k^+)$ with $h(f) = 0$. Then $[\delta_f] = 0$, so $\delta_f = \delta_x$ for some $x = \sum_{i=1}^n \lambda_i e_i + y \in A$ with $y \in \oplus_i(e_i \operatorname{rad} A\, e_i)$ (see Corollary 4.4). Define $g \in C^0(B, I, k^+)$ by $g(e_i) = \lambda_i$ for all $i \in (Q_B)_0$. Then, for any arrow α in Q_B,

$$\delta_f(\alpha(0)) = f(\alpha)\alpha(0) = \delta_x(\alpha(0)) = (\lambda_{t(\alpha)} - \lambda_{s(\alpha)})\alpha(0) + \alpha(0)y - y\alpha(0).$$

Since $\alpha(0)y - y\alpha(0) \in \operatorname{rad}^2 A$ and $\alpha(0) \notin \operatorname{rad}^2 A$, we have $f(\alpha) = \lambda_{t(\alpha)} - \lambda_{s(\alpha)} = d_1(g)(\alpha)$. So $f = d_1(g)$.

(iii) $\widehat{L} \cdot h = 0$ since

$$\widehat{L} \cdot h(f) = \widehat{L}([\delta_f]) = \sum_{m=0}^{c-1} \lambda_m,$$

where for any m with $0 \le m < c$,

$$\lambda_m \overline{p_0 \beta_{p_0}}(m) = \delta_f(\overline{p_0 \beta_{p_0}}(m)) = (f(p_0) - f(p_0))\overline{p_0 \beta_{p_0}}(m) = 0.$$

(iv) Let $[\delta] \in H^1(A)$ be such that $\widehat{L}([\delta]) = 0$. Then $\sum_{m=0}^{c-1} \lambda_m = 0$, where $\delta(\overline{p_0 \beta_{p_0}}(m)) = \lambda_m \overline{p_0 \beta_{p_0}}(m)$. Hence, it follows from Remark 4.10 and Lemma 4.11 that $\sum_{m=0}^{c-1} \lambda^{(i,m)} = 0$ for any $i \in (Q_B)_0$. Let

$$x = \sum_{i \in (Q_B)_0} \sum_{m=1}^{c-1} \left(\sum_{r=0}^{m-1} \lambda^{(i,r)} \right) e_{\nu^m(i)},$$

and let p be any maximal path in kQ_B such that $p = p_- p_+$ with $t(p_-) = i = s(p_+)$. For any m with $0 \le m < c - 1$, we have

$$\begin{aligned}
\delta_x(\overline{p_+ \beta_p}(m)\overline{p_-}(m+1)) &= \overline{p_+ \beta_p}(m)\overline{p_-}(m+1)x - x\overline{p_+ \beta_p}(m)\overline{p_-}(m+1) \\
&= \left(\sum_{r=0}^{m} \lambda^{(i,r)} - \sum_{r=0}^{m-1} \lambda^{(i,r)} \right) \overline{p_+ \beta_p}(m)\overline{p_-}(m+1) \\
&= \lambda^{(i,m)} \overline{p_+ \beta_p}(m)\overline{p_-}(m+1).
\end{aligned}$$

Observe that if $m + 1 = c$, then

$$\begin{aligned}
\delta_x(\overline{p_+ \beta_p}(c-1)\overline{p_-}(c)) &= \overline{p_+ \beta_p}(c-1)\overline{p_-}(c)x - x\overline{p_+ \beta_p}(c-1)\overline{p_-}(c) \\
&= \left(- \sum_{r=0}^{c-2} \lambda^{(i,r)} \right) \overline{p_+ \beta_p}(c-1)\overline{p_-}(c) \\
&= \lambda^{(i,c-1)} \overline{p_+ \beta_p}(c-1)\overline{p_-}(c),
\end{aligned}$$

since $\nu^c(i) = i$ and $\sum_{r=0}^{c-1} \lambda^{(i,r)} = 0$. So

$$\delta(\overline{p_+ \beta_p}(m)\overline{p_-}(m+1)) = \delta_x(\overline{p_+ \beta_p}(m)\overline{p_-}(m+1)).$$

Since $[\delta] = [\delta - \delta_x]$, we may assume that $\delta(\overline{p_+ \beta_p}(m)\overline{p_-}(m+1)) = 0$ for any maximal path p and any m with $0 \le m < c$.

Given any arrow α in Q_B, let $p = p_1 \alpha p_2$ be a maximal path containing α. Then

$$\delta(\overline{\alpha p_2 \beta_p}(m)\overline{p_1}(m+1)) = 0 = \delta(\overline{p_2 \beta_p}(m)\overline{p_1 \alpha}(m+1))$$

and $\overline{p_2 \beta_p}(m)\overline{p_1 \alpha}(m+1) \neq 0 \neq \overline{\alpha p_2 \beta_p}(m)\overline{p_1}(m+1)$ implies that

$$\lambda_{\alpha(m)} + \lambda_{p_2(m)} + \lambda_{\beta_p(m)} + \lambda_{p_1(m+1)} = 0 = \lambda_{p_2(m)} + \lambda_{\beta_p(m)} + \lambda_{p_1(m+1)} + \lambda_{\alpha(m+1)},$$

where $\delta(\overline{p_2}(m)) = \lambda_{p_2(m)}\overline{p_2}(m)$ and $\delta(\overline{p_1}(m+1)) = \lambda_{p_1(m+1)}\overline{p_1}(m+1)$ (see Remark 4.7). So $\lambda_{\alpha(m)} = \lambda_{\alpha(m+1)}$, and therefore $\lambda_{\alpha(m)} = \lambda_{\alpha(0)}$ for any m with $0 \leq m < c$. Since this holds for any arrow α in Q_B we get $\lambda_{\beta_p(m)} = -\lambda_{p(0)}$ for any $p \in \mathbf{P}$ and m with $0 \leq m < c$.

Let f: $(Q_B)_1 \to k^+$ be the function defined by $f(\alpha) = \lambda_{\alpha(0)}$. Hence $\delta = \delta_f$, and we claim that $f \in Z^1(B, I, k^+)$. In fact, given a minimal relation $\rho = \sum_{i=1}^m \lambda_i w_i$, w_1 and w_2 are two paths of positive length in kQ_B sharing starting and terminal points. Then $\overline{w_1} = \mu\overline{w_2}$ for some $0 \neq \mu \in k$ because B is schurian. Suppose $w_1 = \alpha_1 \cdots \alpha_s$ and $w_2 = \beta_1 \cdots \beta_r$. Then $\delta(\overline{w_1}) = \mu\delta(\overline{w_2})$; that is,

$$\sum_{i=1}^{r} \lambda_{\alpha_i(0)}\overline{w_1} = \mu \sum_{i=1}^{s} \lambda_{\beta_i(0)}\overline{w_2}.$$

But $\overline{w_1} \neq 0$ implies that $\sum_{i=1}^r \lambda_{\alpha_i(0)} = \sum_{i=1}^s \lambda_{\beta_i(0)}$. So, by the definition of f, $\sum_{i=1}^r f(\alpha_i) = \sum_{i=1}^s f(\beta_i)$. ∎

COROLLARY 4.13. *Let A be a repetitive covering self-injective algebra, let F: $\widehat{B} \to A$ be a Galois covering with the Galois group $G = (\nu_{\widehat{B}}^c)$, $c \neq 0$, and assume that B is schurian. Then*

$$0 \to \operatorname{Hom}(\pi_1(Q_B, I), k^+) \to H^1(A) \to k^+ \to 0$$

is an exact sequence of abelian groups.

COROLLARY 4.14. *If B is simply connected, then $H^1(A) = k$.*

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REFERENCES

1. H. Asashiba, The derived equivalence classification of representation-finite selfinjective algebras, *J. Algebra* **214**, No. 1 (1999), 182–221.
2. I. Assem, J. Nehring, and A. Skowroński, Domestic trivial extensions of simply connected algebras, *Tsukuba J. Math.* **13**, No. 1 (1989), 31–72.
3. I. Assem and A. Skowroński, On tame repetitive algebras, *Fund. Math.* **142**, No. 1 (1993), 59–84.
4. K. Bongartz and P. Gabriel, Covering spaces in representation-theory, *Invent. Math.* **65**, No. 3 (1981/82), 331–378.
5. O. Bretscher, C. Läser, and Chr. Riedtmann, Self-injective and simply connected algebras, *Manuscripta Math.* **36**, No. 3 (1981/82), 253–307.

6. C. Cibils, Hochschild cohomology algebra of radical square zero algebras, *in* "Algebras and modules, II" (Geiranger, Ed.), pp. 93–101, CMS Conference Proceeding, Vol. 24, Amer. Math. Soc., Providence, 1998.
7. P. Dowbor and A. Skowroński, Galois coverings of representation–infinite algebras, *Comment. Math. Helv.* **62**, No. 2 (1987), 311–337.
8. K. Erdmann, "Blocks of Tame Representation Type and Related Algebras," Lecture Notes in Mathematics, Vol. 1428, Springer-Verlag, Berlin, 1990.
9. K. Erdmann, O. Kerner, and A. Skowroński, Self-injective algebras of wild tilted type, *J. Pure Appl. Algebra* **149**, No. 2 (2000), 127–176.
10. K. Erdmann and A. Skowroński, On Auslander–Reiten components of blocks and self-injective biserial algebras, *Trans. Amer. Math. Soc.* **330**, No. 1 (1992), 165–189.
11. E. Fernández and M. I. Platzeck, Presentations of trivial extensions of finite dimensional algebras and a theorem of S. Brenner, preprint, 1999.
12. P. Gabriel, The universal cover of a representation–finite algebra, *in* "Representations of Algebras (Puebla, Ed.), pp. 68–105, Lecture Notes in Mathematics, Vol. 903, Springer-Verlag, Berlin/New York, 1981.
13. D. Happel, On the derived category of a finite-dimensional algebra, *Comment. Math. Helv.* **62**, No. 3 (1987), 339–389.
14. D. Happel, Hochschild cohomology of finite-dimensional algebras, in "Séminaire d' Algèbre P. Dubreil et M.-P. Malliavin," pp. 108–126, Lecture Notes in Mathematics, Vol. 1404, Springer-Verlag, Berlin, 1989.
15. D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras, *Proc. London Math. Soc. (3)* **46**, No. 2 (1983), 347–364.
16. H. Lenzing and A. Skowroński, On self-injective algebras of Euclidean type, *Colloq. Math.* **79**, No. 1 (1999), 71–76.
17. R. Martínez–Villa and J. A. de la Peña, The universal cover of a quiver with relations, *J. Pure Appl. Algebra* **30**, No. 3 (1983), 277–292.
18. J. A. de la Peña, On the abelian Galois coverings of an algebra, *J. Algebra* **102**, No. 1 (1986), 129–134.
19. Chr. Riedtmann, Representation-finite self-injective algebra of class A_n, *in* "Representation Theory, II," Proceedings of the Second International Conference, Ottawa, 1979, pp. 449–520, Lecture Notes in Mathematics, Vol. 832, Springer-Verlag, Berlin, 1980.
20. A. Skowroński, Selfinjective algebras of polynomial growth, *Math. Ann.* **285**, No. 2 (1989), 177–199.
21. A. Skowroński and K. Yamagata, Socle deformations of self-injective algebras, *Proc. London Math. Soc. (3)* **72**, No. 3 (1996), 545–566.
22. A. Skowroński and K. Yamagata, Galois coverings of self-injective algebras by repetitive algebras, *Trans. Amer. Math. Soc. (2)* **351**, No. 2 (1999), 715–734.
23. T. Wakamatsu, Stable equivalence between universal covers of trivial extensions self-injective algebras, *Tsukuba J. Math.* **9**, No. 2 (1985), 299–316.